The Wheeler-DeWitt equation

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February 10, 2017

Abstract:

The ADM formalism of GR is introduced in the usual 3+1 formalism. We proceed by giving a brief discussion of the background-independent aspect with a bias to a timeless picture. By quantising the ADM formalism we derive at the Wheeler-DeWitt equation(s) and discuss the famous problem of time. At last we go through the guiding lines of the connection formulation and mention the 'Chern-Simons wave function' as a solution to the Wheeler-DeWitt equation.

Main literature

- Kiefer, C. "Quantum gravity (3. Aufl.)." (2012); Ch 3, 4, 5 (& 6).
- Rovelli, Carlo. Quantum gravity. Cambridge university press, 2007; Ch. 2.2

Contents

1 Introduction

In short, we will reformulate general relativity (GR) in the (ADM or) Hamiltonian formalism by showing that the Einstein-Hilbert action results from a Hamiltonian which only consists of two constraints: The diffeomorphism constraint $\mathcal{H}_a \approx 0$ and the Hamiltonian constraint $\mathcal{H}_\perp \approx 0$. After canonically quantising the spatial metric these two take the form $\hat{\mathcal{H}}_a \Psi = 0$ and $\hat{\mathcal{H}}_\perp \Psi = 0$. The former is called the quantum diffeomorphism (or momentum) equation and the latter is named in honour of the work by Wheeler[4] and DeWitt[5] the Wheeler-DeWitt equation.

The idea of quantising GR is one of many attempts to unify two of the most successful theories in theoretical physics. One of these theories is the standard model (SM). As a quantum field theory it describes the fundamental particles to a large success. Even though there are still severe open problems[1] the discovery of the Higgs at the LHC in 2012 and the success of the quark model are a clear indication that a grand unified theory (GUT) must to some extend describe nature in a similar form. The other theory is GR. It has been tested and is tested every day describing nature not only on the astronomical scales but as the recent detection of the gravitational waves show also on much smaller scales (of gravitational interaction). In fact this detection presents a first test in the very-strong field regime of GR and provides further evidence for the existence of black holes. Up to date it is an open question how to combine these two theories to a consistent description of nature at all scales.

Even though the brute force quantisation of GR presented in the following sections is quite a drastic upshot and brings with it numerous problems, it lead to a connection representation of GR, which ultimately lead to the development of quantum loop gravity (QLG). Which is depending on whom one asks next to or after string theory one of the furthest developed GUT at our hands.

2 The ADM formalism

A revelation of the (ADM[2] or) Hamiltonian formulation of GR is the separation of the Einstein Field equations $G^{\mu\nu} + \Lambda g^{\mu\nu} = T^{\mu\nu}$ into the time-independent equations of (or constraints on) the metric from the time dependent (or evolution equations) of the metric. This way the constraints describe the data one can encode on the metric and the evolution equations their evolution w.r.t time. One speaks of the *initial value formalism* of GR. We will come back to this point after deriving the constraints of GR. Bear in mind that for a canonical quantisation of GR the Hamiltonian description is needed to substitute the Poisson bracket with the commutator.

2.1 Foliation of spacetime and the ADM action

In the ADM formalism of GR spacetime, a $3+1$ dimensional Riemannian manifold $\mathcal{M}$ with metric g, is *foliated*[3] into a one parameter family or trajectory of space-like slices (*leaves*[3]) $\{\Sigma_t | t \in \mathbb{R}\}$ (see Figure 1). Unless otherwise noted we will assume the spatial slices to be closed[4].
Each of these slices has the dynamical (ADM) variables:

- h_{ab} the metric tensor of the spatial slices induced by g (pullback of the ambient spacetime)
- p^{ab} their conjugate momenta (w.r.t. the poisson bracket, i.e. $\{h_{ab}(\mathbf{x}), p^{cd}(\mathbf{y})\} = \delta^c_{(a}\delta^d_{b)}\delta(\mathbf{x},\mathbf{y})$)

Note that we will only introduce the '*distinguished time*', induced by this background foliation, for the moment being. On the one side of the coin we need this 'distinguished time' in order to define a conjugate momentum, $p = \frac{\partial L}{\partial \dot{q}}$ and work in an Hamiltonian formalism. While on the other side of the coin GR is *background-independent*. This means that there is no background structure, 'everything is relative' or in this context: there is no absolute time. We will come back to the discussion of background-independence in the Hamiltonian formalism after deriving it.

[1] Hierachy problem, strong CP problem, ...

[2] After Richard Arnowitt, Stanley Deser and Charles W. Misner[6]

[3] see mathematically precise definition

[4] In the open case one can attribute asymptotic boundary conditions. These take the form of asymptotic boundary conditions with respect to (w.r.t.) an embedding into another manifold. However, such conditions imply that there is a part of the universe outside the region being modeled and since we consider GR to be a theory of the whole universe we shall restrict ourselves to the closed case.

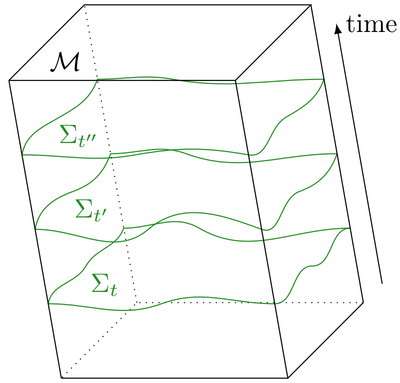

Figure 1: A foliation of a manifold $\mathcal{M}$ into spatial slices Σ_t

In order to construct a Hamiltonian formalism we will not start with the tensorial form of the Einstein Field equations $G^{\mu\nu} = T^{\mu\nu}$ but the Einstein-Hilbert action

$$16\pi G_N S_{EH} = \int_{\mathcal{M}} dt d^3x \ \sqrt{g}\big(R - 2\Lambda\big).$$

With the background-independence of GR in mind we will express this action only in the ADM variables of the spatial slices, i.e. eliminate the time components[5].

The heart of this step is Gauß famous *theorema egregium* (remarkable theorem) in $3+1$ dimensions. The remarkable part of this theorem is that one can express the extrinsic curvature R of the spatial slice Σ_t given by the embedding spacetime $\mathcal{M}$ only by intrinsic parameters, i.e. those which can be determined within Σ_t:

$$R = K_{ab}K^{ab} - K^2 + {}^{(3)}R. \tag{1}$$

${}^{(3)}R$ is here the induced curvature within the spacial slice and $K_{ab}(\mathbf{x}) = \frac{1}{2}\mathcal{L}_{\mathbf{n}} h_{ab}(\mathbf{x})$ the Lie derivative of h_{ab} w.r.t. the (time-like) unit vector field $n^\mu(\mathbf{x})$ orthogonal to the slices.

One can show that there always exists a globally hyperbolic '*time function*' $t^\mu(\mathbf{x})$, i.g. not orthogonal to the slices, for the foliation such that each slice is a Cauchy surface (constant in time t^μ; each point on the slice has the same time value). In other words the time arrow in Figure 1 just indicated the overall direction. The actual vector field $t^\mu(\mathbf{x})$ can take more arbitrary flows (see Figure 2).

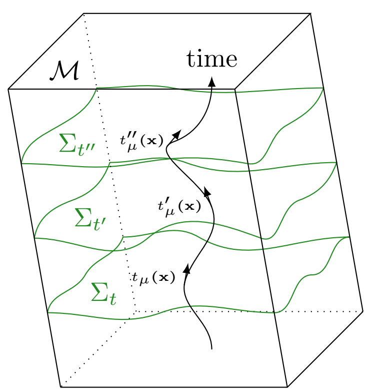

Figure 2: An illustration of the 'time function' $t_\mu(\mathbf{x})$ moving through a foliated manifold $\mathcal{M}$ for a fixed point $\mathbf{x}$.

Let us decompose this flow of time into components normal ($n^\mu n_\mu = -1$) and tangential to Σ_t,

$$t_\mu(\mathbf{x}) = N(\mathbf{x})n_\mu(\mathbf{x}) + N_\mu(\mathbf{x}).$$

We can see in Figure 3 that $N(\mathbf{x})$ is the relative factor between the (absolute) time of the embedding space and the induced time experienced by moving though the foliation and $N^a(\mathbf{x})$ describes how a fixed point in the spatial slice Σ_t is shifted in the embedding space to the same point in the next slice Σ_{t+dt}. In short, the *lapse function* $N(\mathbf{x})$ and the *shift function* $N^a(\mathbf{x})$ describe how to move from slice to slice. In this sense we say that:

[5] By tracing the second Bianchi identity accordingly one can show that the 0ν components of the Einstein Field equations do not determine the time evolution of the metric. In other words, we expect the 4 Einstein Field equations corresponding to the 0ν and $\nu 0$ entries to describe the constraints and the other 6 spatial entries to describe the time evolution. We can take this as a motivation to separate time from the spatial slices.

– N_a generates spatial diffeomorphsims while ‘gliding’ through the foliation

– N generates pointwise change in the velocity with which one ‘glides’ through the foliation

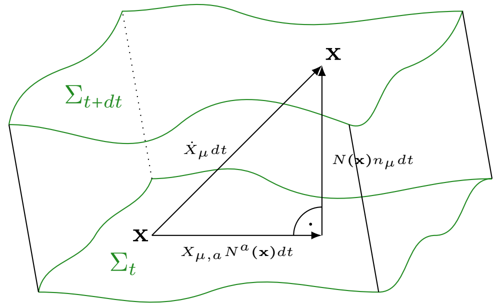

Figure 3: An illustration of how the ‘time function’ $t_\mu(\mathbf{x})$ maps points from one spatial slice Σ_t to the spatial slice of the next instance Σ_{t+dt}. X_μ are the coordinates of the embedding spacetime $\mathcal{M}$.

One can express the volume element $\sqrt{g}$ in terms of the induced metric $\sqrt{h}$ (see [1, Ch. 4])

$$\sqrt{g} \;=\; N\sqrt{h}. \tag{2}$$

After inserting Equation 1 and Equation 2 in the Einstein-Hilbert action we get the ADM action:

$$16\pi G_N S_{ADM} \;=\; \int_{\mathcal{M}} dtd^3x\;\; \overbrace{N\sqrt{h}}^{\sqrt{g}}\Big(\overbrace{K_{ab}K^{ab} - K^2 + {}^{(3)}R - 2\Lambda}^{R\text{ (thm. egregium)}}\Big)$$

One may think of K_{ab} as the velocity field of $h_{ab}(\mathbf{x})$ ‘gliding’ through the foliation. By defining the ‘DeWitt metric’ $G^{abcd} := \frac{\sqrt{h}}{2}\big(h^{ac}h^{bd} + h^{ad}h^{bc} - 2h^{ab}h^{cd}\big)$ this action takes the form

$$16\pi G_N S_{ADM} \;=\; \int_{\mathcal{M}} dtd^3x\;\; N\Big(\underbrace{G^{abcd}K_{ab}K_{cd}}_{\text{‘kin. term’}} + \underbrace{\sqrt{h}\left({}^{(3)}R - 2\Lambda\right)}_{\text{‘pot. term’}}\Big). \tag{3}$$

The contribution of the ‘kinetic’ term to the action is determined by the ‘DeWitt metric’ G^{abcd}. We will postpone its discussion and insert the exact expression for $K_{ab} = \frac{1}{2N}\left(\dot{h}_{ab} - D_aN_b - D_bN_a\right)$, where D_b is the covariant derivative[6] on Σ, into the ADM acton to rewrite it in the form[7]:

$$16\pi G_N S_{ADM} \;=\; \int_{\mathcal{M}} dtd^3x\;\Big(\overbrace{\underbrace{p^{ab}\dot{h}_{ab}}_{\text{‘}pv\text{’}}\underbrace{-N\mathcal{H}^g_\perp - N^a\mathcal{H}^g_a}_{-\mathcal{H}^g\text{ (‘constraints’)}}}^{\mathcal{L}}\Big) \tag{4}$$

with

$$\begin{aligned}\mathcal{H}^g_\perp &= 16\pi G_N G_{abcd}p^{ab}p^{cd} - \frac{\sqrt{h}}{16\pi G_N}\left({}^{(3)}R - 2\Lambda\right)\\ \mathcal{H}^g_a &= -2D_b{p_a}^b.\end{aligned}$$

It should be worth mentioning that in the presence of Yang-Mills fields $\mathcal{H}^g_\perp$ gets an additional contribution $H^{YM}_\perp$, i.e. $\mathcal{H}^g_\perp \to \mathcal{H}^g_\perp + H^{YM}_\perp$ (see [1, Ch. 4.1] for a derivation).

2.2 Recovering Background-independence

The diffeomorphism- and Hamiltonian constraint

Recall that GR is background-independent and therefore the information of the (fixed background) foliation encoded in the *lapse-* and *shift functions* N and N^a can be arbitrary functions. By varying Equation 4 w.r.t to N or N^a and

[6] $D_aT^{b_1\dots b_r}{}_{c_1\dots c_s} = h^d_a\; h^{b_1}_{e_1}\dots h^{b_r}_{e_r}\; h^{c_1}_{f_1}\dots h^{c_s}_{f_s}\;\nabla_dT^{e_1\dots e_r}{}_{f_1\dots f_s}$ (h^d_a ‘projects onto’ Σ)

[7] For a step-by-step discussion see [1, Ch. 4]

demanding that the expression vanishes, we get the *Hamiltonian-* or *momentum* (or *diffeomorphism*) constraint:

$$\begin{aligned} \mathcal{H}^g_\perp &\approx 0 \quad \text{(Hamiltonian constraint)} \\ \mathcal{H}^g_a &\approx 0 \quad \text{(momentum- or diffeomorphism constraint).} \end{aligned}$$

This ensures, independently on how N and N^a vary (i.e. the spacial slices evolve in time), that these constraints and thus the Einstein Field equations be fulfilled at all events. And just as important, it ensures that we recover the background-independence of GR, since N and N^a are now the Lagrange multipliers of the constraints.
With this in mind let us carefully distinguish between the foliation $\{\Sigma_t | t \in \mathbb{R}\}$ of some Riemannian manifold $\mathcal{M}$ and the spatial slice Σ we are evolving in time by mapping it into or perhaps in a more intuitive sense 'gliding though' this foliation (see Figure 4). It is crucial to arrive at this distinction to recover the background-independent picture.

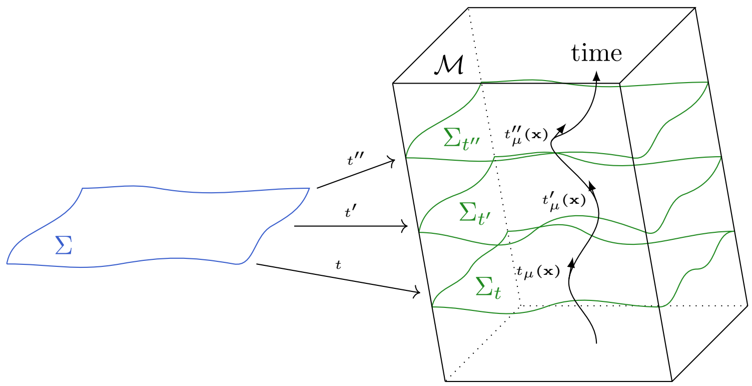

Figure 4: Mapping a spacial slice Σ through a foliation of a manifold $\mathcal{M}$

Time in the ADM formalism

The ADM formalism is by construction in the 3+1 decomposition of GR and therefore its time evolution as a Hamiltonian system is given by the Poisson bracket and the Hamiltonian

$$\frac{dF(\mathbf{x})}{dt} = \Big\{F(\mathbf{x}), \mathcal{H}^g(\mathbf{x})\Big\}_{\text{P.B.}} + \frac{\partial F(\mathbf{x})}{\partial t}.$$

This, however, is the evolution of 'gliding' through the foliation and therefore not background-independent. More precisely, we saw in the previous subsection that in this Hamiltonian form of GR the *choice of time* is fixed by '*pinning down*' points to any background structure. To regain background-independence we separated the spatial slice Σ from foliation of some manifold.

Now we are left with the problem of finding a background-independent way of recovering a time for our theory. In this sense the demand for background-independence in the ADM formulation reveals a *timeless*[8] aspect or redundancy of the choice of time at the classical level.

In the attempt to find a possible solution to this problem we will first draw a parallel between the ADM formulation and electrodynamics as initial value formalisms and then see to what extend GR is a relational theory[9]. The latter will show us whether one can reconstruct, given the right information, a time form the three-dimensional geometry of two configurations (slices).

Time in a relational system

The Hamiltonian of GR consisted only on the (time-independent) Hamiltonian- and momentum constraints. The corresponding equations in classical electromagnetism (EM) are the time-independent Maxwell equations. Likewise the time evolution of the foliation corresponds to the time-dependent equations[10].

[8]For aspects of timelessness in classical dynamics see the beautifully written essay of J. Barbour [8] in which he is surprisingly pleased with the idea of a timeless theory. For a more detailed and closer discussion w.r.t. GR see [1, Ch. 3.4].

[9]Space or time is defined as relationships among events.

[10]Note that one can use the 4 degrees of freedom (d.o.f.) of the constraints to fix 4 of the 6 d.o.f. of the evolution equations of the spatial slices. The remaining 2 d.o.f. can in the transverse traceless gauge be identified with the x and $+$ polarisations of the weak-field expansion of the gravitational field.

	ADM formulation of GR	classical EM
time independent eq. of motion (hold at all times)	$\mathcal{H}_\perp$ and $\mathcal{H}_a$ (4 d.o.f.)	$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$ $\vec{\nabla}\cdot\vec{B} = 0$
time dependent eq. of motion (evolution in time)	evolution of h_{ab} ('gliding' through the foliation) (6 d.o.f.)	$\dot{\vec{E}} = +\vec{\nabla}\times\vec{B} - 4\pi\vec{j}$ $\dot{\vec{B}} = -\vec{\nabla}\times\vec{E}$

In electrodynamics specifying the B-field on two hypersurfaces is not sufficient, additionally one needs to fix the time parameters of the two hypersurfaces. Therefore, in the case of our timeless version of the ADM formulation we would expect additionally to the configurations of two spatial slices (Cauchy surfaces) the corresponding times. With this in mind, there is the so called '*sandwich conjecture*' (see Figure 5), which states that one can always determine the temporal separation between two slices. There is also the (stronger) infinitesimal version of this conjecture, the '*thin-sandwich conjecture*' (formulated by [7]), which only needs the spatial metric and the infinitesimally propagating direction in phase space to determine the temporal evolution of h_{ab}.

But is the 'sandwich conjecture' enough to predict the future evolution of the slices? There is not much known. However, in the classical Newtonian case one cannot predict the future evolution by two such successive 'pictures of the universe[11]', since one lacks the information of the angular momentum and kinetic energy of the system. This is the so called '*Poincaré defect*'[10]. A way around this problem has been developed by Barbour and Bertotti [11] by introducing a 'gauge freedom' w.r.t. translations and rotations and then 'fixing this gauge' (see [1, Ch. 3.4]). We will see an analog in going to the connection representation of GR by replacing the metric field with a *dreibein* invariant under rotations. But lets not get ahead of ourselves.

If the two times of these configurations or its the relative speed is given and we neglect the 'Poincaré defect', one can determine the future evolution from the two configurations in the relational system. Barbour goes a step further and argues for a completely timeless model in which 'time can be read off the heaven' by taking successive snapshots of the universe. For a classical example see his essay [8]. He pursuits to develop a completely timeless description of the universe in which only 'shapes[12]' physically matter. Its important to stress that this is only one of many speculative directions and not widely accepted.

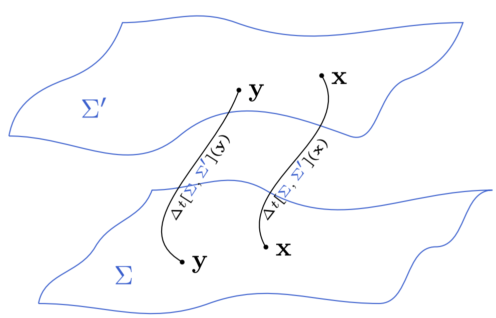

Figure 5: Illustrating the 'sandwich conjecture'.

The question whether space is an absolute- or relational structure dates back to Aristotle, Descartes, Newton[13], Leibnitz, Mach and many more. But is GR itself a relational system? Not completely, only when we mod out the diffeomorphisms of GR a relational physical theory is recovered [9]. W.r.t. the question of background-independence it is worth having a closer look at Einsteins struggle with the *general covariance* of GR in Appendix A (we closely followed [3, Ch. 2.2.5]).

In the ADM formalism our parameter space consisted of the foliation and the ADM variables. Since the momentum was defined using the distinguished time of the foliation our parameter space of the background-independent timeless picture only consists of the set of spatial (pseudo-)Riemannian metrics Riem(h_{ab}). In the coming subsection 2.3 we will identify the metrics related by diffeomorphisms and discuss the resulting quotient space, called 'superspace'. Let us now summarise our attempt so far to recover background-independence from the ADM formalsm.

[11] consisting of a collection of point-masses

[12] Angles in triangles, matter ratios,... (see http://www.platonia.com/research.html)

[13] Part of the critique of Newtons concept of space as an absolute structure comes from his introduction of an absolute space. Furthermore, for Newton time as a parameter was introduced for the purpose of solving his differential equations. In this sense it was a mathematical trick. However, Einsteins development of GR introduced a background-independent theory. For an in depth discussion we will direct the reader to Rovelli's wonderful book [3, Ch. 2.2 especially Ch. 2.2.2 & 2.2.5], once again Barbours essay [8], [1, Ch. 3.4] or [9].

Summary

We started reconstructing the background-independent picture form the ADM action by varying the lapse- and shift-function. Their resulting equation of motions (eom) are called the Hamiltonian- and momentum-constraint and completely made up the Hamiltonian of GR. Consequently, we had to distinguish between the spatial slices of the background-manifold and the separated spatial slice which we evolved through the foliation. Arriving at a what appears to be timeless three-dimensional picture.

Motivated by Einsteins struggle with the general covariance of GR, we proceeded by identifying diffeomorphic spatial slices. This way we retrieved GR as a timeless relational system which is still background-independent. In such a system time-evolution can be recovered by 'taking successive pictures of the universe', i.e. two spatial slices, provided that we have, in analogy to the electromagnetic example, the times of both (Cauchy surfaces) and neglect the 'Poincaré defect'. In order to not to go down into too recent and speculative paths we left the reader at this point.

2.3 Superspace and the 'De-Witt metric'

Superspace

In order to gain a relative system we argued in the previous subsection to mod out the diffeomorphisms of our spatial slice Σ. Therefore instead of considering the whole configuration space[14] of spatial (pseudo-)Riemannian metrics h_{ab} of Σ we identify metrics related by a diffeomorphism and form the quotient space:

$$\mathcal{S}(\Sigma) \quad := \quad {}^{\mathrm{Riem}(\Sigma)}\!/_{\mathrm{Diff}(\Sigma)}.$$

This space was introduced by Wheeler as *superspace*[15].

The 'De-Witt metric'

In fact, the 'De-Witt metric' we introduced in Equation 3 plays the role of a metric on $\mathrm{Riem}(\Sigma)$. Due to its symmetry it can be considered to be a 6×6 matrix which maps two (symmetric) mertrics l_{ab}, k_{cd} tangent to a point h_{ab} in $\mathrm{Riem}(\Sigma)$ to a real number.

$$G(l,k) \quad := \quad \int_\Sigma d^3x \;\, G^{abcd} l_{ab} k_{cd}.$$

It might be useful to note that the 'DeWitt-metric' has the same symmetries as an elasticity tensor. After diagonalisation one finds that it is of Lorentzian signature

$$(-,+,+ \;\;,\;\; +,+,+),$$

which makes it an indefinite metric. Recall that the 'De-Witt metric' describes the kinematic term of the ADM Lagrangian (Equation 3), consequently one can show that its indefinite sign determines whether gravity acts as a attractive- or repulsive force (see [1, Ch. 4.2.5]). Therefore, this metric governs the dynamics of GR in the ADM formalism.

It is worth pointing out, in the premises of the connection representation of GR, that if we would use left-handed triads instead of metrics, the minus sign would necessarily flip one left-handed triad to a right handed one. A metric doesn't keep track of such handedness.

So far so good, but is the metric also a metric on superspace? In superspace we cannot distinguish between the 'vertical' direction along the orbits of diffeomorphic metrics and the 'horizontal' direction orthogonal to it (see Figure 6). This is where the indefinite property of the 'De-Witt metric' becomes troublesome, since in the 'light-like' case of zero norm the vertical- and horizontal ('light- and space-like') subspaces overlap. Consequently, the 'De-Witt metric' is only a metric on the 'time- and space-like' sectors.

[14] Additionally one can include the foliation of a background manifold in the configuration space and the configuration space of the matter fields.

[15] Not to be confused with the superspace of supersymmetry (SUSY).

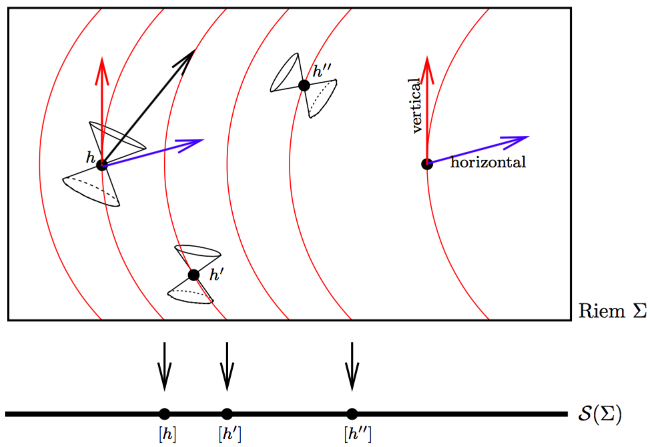

Figure 6: An illustration (taken from [1, Ch. 4.2]) of projecting orbits of diffeomorphic metrics in Riem(Σ) to points in superspace. The cones indicate the Lorentzian 'De-Witt metric'. h lies in the time-like sector, h' on the light-like sector and h'' in the space-like sector. The point $[h']$ lies on a border between the time- and space-like sectors in superspace. The 'De-Witt metric' is not a metric at points of such borders.

Propagating in superspace instead of Riem(Σ) ensures that we only propagate between different physical solutions of the background-independent formalism of GR. Loosely speaking we modded out the 'gauge group' of GR and are left with such 'light- and space-like' sectors. Again, to recover background-independence, as we tried in the previous section, we are left with the timeless picture. One could either keep a timeless picture or 'fix a time' either before or after quantisation. Note that time evolution doesn't necessarily imply to 'propagate from one point in superspace to another'. Instead time evolution may also be realised through a topological change of superspace (see Figure 7). But, topology-changing spacetimes come already at the classical level with severe problems such as singularities, closed time-like curves and degenerate metrics[2]. Consequently, there is much room for speculation and we are left again with a questionable proposition.

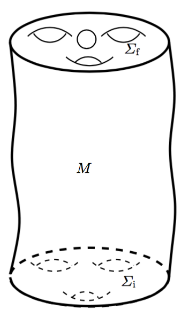

Figure 7: An illustration of 'growing a nose' on a superspace Σ_i (taken from [2]). Note that, the structure of the superspace $\mathcal{S}(\Sigma)$ is in general of a more complicated form than Riem(Σ). Furthermore, recall that we restricted ourselves to compact slices.

3 The Wheeler-DeWitt equation

If we knew what it was we were doing,
it would not be called research, would it?

- Albert Einstein

The quantisation of an Hamiltonian system is at the mathematical level a straight forward procedure: In the case of the Hamiltonian formalism of GR we promote h_{ab} and p^{cd} to operators

$$\begin{aligned} \hat{h}_{ab}(\mathbf{x})\Psi[h_{ab}(\mathbf{x})] &= h_{ab}\Psi[h_{ab}(\mathbf{x})] \\ \hat{p}^{cd}(\mathbf{x})\Psi[h_{ab}(\mathbf{x})] &= \frac{\hbar}{i}\frac{\delta}{\delta h_{cd}}\Psi[h_{ab}(\mathbf{x})] \end{aligned}$$

and replace the Poisson bracket with the usual commutation relations

$$\left[\hat{h}_{ab}(\mathbf{x}), \hat{p}^{cd}(\mathbf{y})\right] = i\hbar\delta^c_{(a}\delta^d_{b)}\delta(\mathbf{x}, \mathbf{y}).$$

The highly non-trivial part is actually make sense of what we are doing. $\Psi[h_{ab}(\mathbf{x})]$ is here a wave functional and its not clear where its living in. We will implement the constraints according to Dirac:

$$\hat{\mathcal{H}}_\perp \Psi[h_{ab}(\mathbf{x})] \;:=\; \Big(-16\pi G_N \hbar^2 G_{abcd} \frac{\delta^2}{\delta h_{ab} \delta h_{cd}} - \frac{\sqrt{h}}{16\pi G_N} \left({}^{(3)}R - 2\Lambda \right) \Big) \Psi[h_{ab}(\mathbf{x})] = 0$$
$$\hat{\mathcal{H}}_a \Psi[h_{ab}(\mathbf{x})] \;:=\; \Big(-2 D_b H_{ac} \frac{\hbar}{i} \frac{\delta}{\delta h_{bc}} \Big) \Psi[h_{ab}(\mathbf{x})] = 0.$$

The first of these two functional equations is called the Wheeler-DeWitt equation and the second is called the quantum diffeomorphism equation. Sometimes one refers to both as the the 'Wheeler-DeWit equations'. There are many problems[16] with regard to the Wheeler-DeWit equations. A central one is that one can not define[1, Ch. 5.2.2] a suitable measure for a Schrödinger-type inner-product

$$\langle \Psi_1 | \Psi_2 \rangle \;=\; \int_{\mathrm{Riem}(\Sigma)} \mathcal{D}\mu[h] \;\; \Psi_1^*[h] \Psi_2[h].$$

Hence, such a construction is at most formal. Also note that the 'DeWitt metric' explicitly appears in the Wheeler-DeWitt equation. Therefore the discussion of superspace is also vital after canonical quantisation.

We will now turn to the famous *problem of time* in quantum gravity (QG) and then proceed to the connection representation.

The problem of time

In quantum mechanics (QM) time is an absolute parameter governing the evolution of states:

$$\hat{\mathcal{H}} \Psi \;=\; i\hbar \frac{\partial}{\partial t} \Psi.$$

But GR is background-independent and cannot depend on an absolute time. In fact, we demand by background-independence that the l.h.s., the Hamiltonian, vanished upon acting on the wave functional (for an empty universe). A consistent theory of QG should therefore exhibit a novel concept of time. This is called the *problem of time*. As mentioned before, to fix this problem we are left with the cases of considering a timeless model, fix a time before- or after quantisation. There is much speculation concerning each of these paths. At last we once again point out that the role of time in QG is not clear. Quoting Smolin[9]:

> '*The issue [the problem of time] is controversial and there is strong disagreement among experts. [...] The problem of time is a key challenge that any complete background independent quantum theory of gravity must solve.*'

In the case of our canonical quantisation of the ADM formalism, time and therefore spacetime is absent in the 'Wheeler-DeWitt equations'. Only the spatial metric has survived in the argument of the wavefunctional, whose embedding space is not clear. Time as well as probability are both left in this formalism as phenomenological concepts.

A way to think of this phenomenon is to draw the analogy to the disappearance of trajectories in quantum mechanics that time here disappeared from space-time. From this perspective the wavefunctional would smears out metrics in superspace, in the same sense as it usually *smears out* paths in space. Consequently, a sum over paths would corresponds to a sum over histories.

Change of parameters

To solve the central Wheeler-DeWitt equation presented above three choices of variables have been developed:

- Quantum-Geometrodynamics with variables: three-metric and the extrinsic curvature
- Connection dynamics with variables: a connection A_a^i and a non-abelian electric field E_i^a
- Loop dynamics: variables connected by holonomies with A_a^i and the Flux of E_i^a

We will now briefly present the connection formulation of GR and comment on a solution of the corresponding quantum Wheeler-DeWitt equations.

[16] For a overview see [1, Ch. 5.2].

4 The connection formulation of GR

The Ashtekar variables

In the connection formulation of GR we replace the usual gravitational field of the ADM formalism[17] given by the (spatial-)metric $g_{ab}(x)$ with a *triad* (*Dreibein*)

$$E_i^a(x) := \sqrt{h(x)} e_i^a(x). \tag{5}$$

with $\sqrt{h(x)} = |\det(e_a^i)|$. By using these so called *Ashtekar variables* we added a freedom or orientation and handedness[18] to the gravitational field which varies from point to point in space (see Figure 8). In other words, using the Ashtekar variables 5 the gravitational field takes the form of a non-abelian vector field with gauge-group $SO(3)$ or its double cover $SU(2)$ (with an additional handedness). To promote this additional gauge freedom to a local property we introduce additionally to the Levi-Civita connection $\Gamma^i{}_{jk}$ the corresponding spin connection $\omega_a{}^i{}_j = \Gamma^i{}_{jk} e_a^k$ of the gauge-group

$$D_a e_b^i = \partial_a e_b^i - \Gamma^c{}_{ab} e_c^i + \omega_a{}^i{}_j e_b^j.$$

From the orthogonality condition of the triad:

$$h_{ab} e_i^a e_j^b = \delta_{ij}$$

one can recover the local (inverse) metric

$$h^{ab} = e_i^a e_i^b.$$

One may think of the local rotational d.o.f. of the triad as an intrinsic spinorial d.o.f. of a field. In fact, when Ashtekar formulated this connection representation he was influenced by Penroses Twistor theory and therefore first chose the gauge group $SU(2)$. In this comparison the spin corresponds to the orientation of the triad in the embedding space.

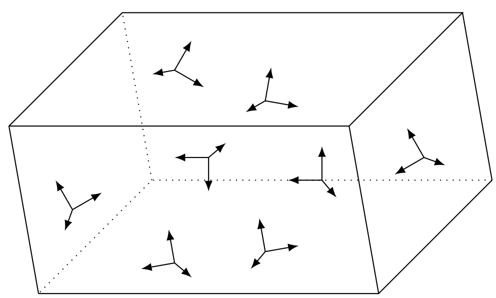

Figure 8: Illustrating triads in spacetime. The reader should be aware that the local triad is the gravitational background. In this sense this picture is not background-independent. We already arrived at a formulation which are not fields in spacetime, but fields on fields.

Analogously to the familiar spinorial case we define the object:

$$\Gamma_a^i := -\frac{1}{2} \omega_{ajk} \epsilon^{ijk},$$

which describes the rotation $\delta\omega^i$ of parallel transporting a vector v^i (see Figure 9) through

$$\begin{aligned} \delta\omega^i &= \Gamma_a^i dx^a \\ dv^i &= \epsilon^i{}_{jk} v^i \delta\omega^k. \end{aligned}$$

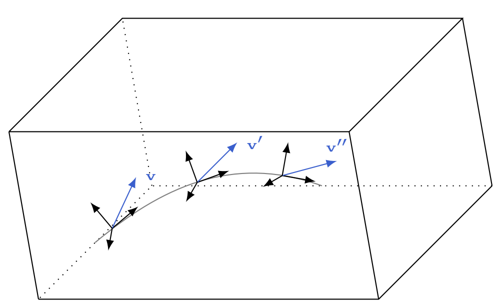

Figure 9: Parallel transport of a vector along a rotating triad.

[17] There is also a *vierbein* formalism for the common formulation, see [3, Ch. 2]
[18] Left- or right-handed field

Let us proceed by introducing a new variable

$$K_a^i \quad := \quad K_{ab} e^{bi}.$$

K_{ab} is here the second fundamental form of the spatial slice. One can shown that

$$K_a^i \delta E^{ia} = -8\pi G_N p^{ab} \delta h_{ab}.$$

in other words K_a^i is the canonically conjugate momentum of E_i^a. By going to the triad formulation we ended up with 9 d.o.f. for each triad and need to fix the gauge this is done by imposing the constraint

$$\epsilon_{ijk} K_a^j E^{ka} \approx 0.$$

It is called as the *Gauß constraint* and ensures that the angular momentum vanishes '$\mathbf{p} \times \mathbf{x} = 0$'. The 9 d.o.f. of the triad e_i^a are now reduced to the 6 d.o.f. of h_{ab}.

The connection A_a^i

Ashtekars central step consisted in combining E_i^a and K_a^i into a complex connection A_a^i

$$G_N A_a^i \quad = \quad \Gamma_a^i + \beta K_a^i,$$

where β is the 'Barbero-Immirzi' parameter. The crucial property of our new connection A_a^i is that it is the canonically conjugate variable of $E_j^b/8\pi\beta$:

$$\left\{A_a^i(\mathbf{x}), E_j^b(\mathbf{y})\right\}_{\text{P.B.}} \quad = \quad 8\pi\beta \delta_j^i \delta_a^b \delta(\mathbf{x}, \mathbf{y}).$$

In addition one has:

$$\left\{A_a^i(\mathbf{x}), A_b^j(\mathbf{y})\right\}_{\text{P.B.}} \quad = \quad 0.$$

The associated curvature is

$$F_{ab}^i \quad = \quad 2G_N \partial_{[a} A_{b]}^i + G_N^2 \epsilon_{ijk} A_a^j A_b^k.$$

The Hamiltonian constraints take for $\beta = i$ and $\Lambda = 0$ the form

$$\tilde{\mathcal{H}}_\perp \quad = \quad \frac{1}{2\sqrt{|\det(E_i^a)|}} \epsilon^{ijk} F_{abk} E_i^a E_j^b \tag{6}$$

$$\tilde{\mathcal{H}}_a \quad = \quad F_{ab}^i E_i^b \approx 0 \tag{7}$$

These have to hold additional to the Gauß constraint. However, for other choices there is an additional term in the Hamiltonian constraint. In fact, black holes seem to prefer a particular value. One can fix the 'Barbero-Immirzi' parameter β by demanding to get the Beckenstein-Hawking limit for the entropy of a black hole[12], the result depends on the gauge group. Numerical simulation have been performed[13] with the same task.

The 'Chern-Simons solution'

After quantization w.r.t our new connection one can reformulate the Wheeler-DeWitt equation:

$$\epsilon^{ijk} \frac{\delta}{\delta A_a^i} \frac{\delta}{\delta A_b^j} \left(F_{kab} - \frac{i\hbar\Lambda}{6} \epsilon_{abc} \frac{\delta}{\delta A_c^k} \right) \Psi[A] \quad = \quad 0.$$

The exact solution by Kodama[14] given a nonzero cosmological constant in the form of a '*Chern-Simons wavefunction*'

$$\Psi_\Lambda \quad = \quad \exp\left(i \frac{6}{G_N \Lambda \hbar} S_{CS}[A] \right),$$

with the '*Chern-Simons action*'

$$S_{CS}[A] \quad = \quad \int_\Sigma d^3x \; \epsilon^{abc} \mathrm{Tr} \left(G_N^2 A_a \partial_b A_c - \frac{2}{3} G_N^3 A_a A_b A_c \right).$$

This state is both gauge- and diffeomorphism invariant, but not normalisable[1, Ch. 6.1.1]. They, however seem to lie in the subspace of physical solutions which we did not discuss. In the case of vanishing cosmological constant some solutions of the Wheeler-DeWitt equation have been expressed in therms of knot invariants [15].

A Einsteins “hole” argument

Einsteins “hole” argument After the development of special relativity Einstein, in the race with Hilbert for the field equations of GR, long struggled with the *concept of general covariance*[19]. Einsteins argument against it was his so called *“hole” argument*: Consider a part of space-time with no matter content, then the gravitational field at two different points in this “hole” may have a turbulent gravitational field at point A and flat gravitational field at point B. Now, consider a smooth map which reduces to the identity outside the “hole” and maps the turbulent point A to the flat point B. Then both would be solutions to the field equations and have the same past, but Newtons classical theory was deterministic. This let Einstein initially to panic, in order to get the right classical limit, he concludes that either the field equations would not be generally covariant or there is no meaning in referring to point A. At last, he observes that the ‘turbulent points’ where particles starting from the same field solution meet will be mapped under a diffeomorphism to another point with the same ‘turbulent points’ configuration (see Figure 10).

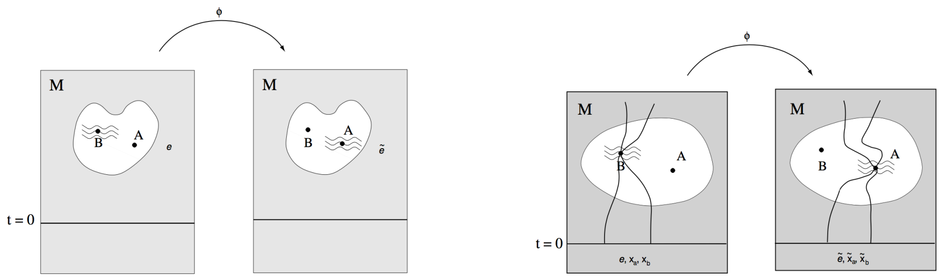

Figure 10: Illustrations of Einsteins “hole” argument (taken from [3, Ch. 2.2.5]).

Simply put, general covariance can be recovered if instead of asking ‘*What is the field configuration at a specific point of our theory?*’ we ask ‘*What is the field configuration at the point where two particles meet/interact?*’. This way general covariance is recovered. Note that the first question labels or ‘pins down’ spacetime points, while the latter only ‘pins down’ points of the interaction points of matter fields in relation with each other. The latter is background-independent and does not need a background manifold.

Einstein concludes that the theory does not predict what happens at spacetime points, but rather that it predicts what happens at locations determined by the dynamical elements of the theory themselves[20]. He calls such points *spacetime coincidences.*

The gravitational field is therefore not just a local background but a field in itself. Motion is now entirely relative to two field configurations. No more fields in spacetime. Spacetime is a generally covariant and relative structure. We now have “fields on fields”. Localising or ‘pinning down’ points on a background manifold has in this picture no physical meaning. Furthermore, both configurations of the gravitational field are physically indistinguishable. In this sense the group of space-time diffeomorphism is part of the gauge group of GR. Einstein developed a consistent background-independent picture of GR by modding out the diffeomorphisms and going to a purely relational system.

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