CHAPTER 6

EINSTEIN EQUATIONS

> You will be convinced of the general theory of relativity once you have studied it. Therefore I am not going to defend it with a single word.
>
> A. Einstein

6.1 The energy-momentum tensor

Having decided that our description of the motion of test particles and light in a gravitational field should be based on the idea of curved space times with a metric, we must now complete the theory by postulating a law to say how the sources of the gravitational field determine the metric. To construct this gravitational field equation we must first find a covariant way of expressing the source term ρ in the Poisson equation

$$\nabla^2\Phi = 4\pi G\rho\,. \tag{6.1}$$

It is clear that the relativistic generalization of Eq. (6.1) cannot simply involve ρ as the source of the relativistic gravitational field, since ρ is the energy density measured by only one observer, that at rest with respect to the fluid element. It is not the first time that we find this kind of situation. The relativistic formulation of Maxwell equations needed the combination of the charge density ρ_e and the charge current J^i into a 4-vector $J^\mu = (\rho, J^i)$ with the right transformation properties. Can we do something similar here? The most naive trial would be a combination of the energy density ρ with some energy flux s^i into a 4-vector, let's say $s^\mu = \left(\rho, s^i\right)$. However, the total energy in this case, $E = \int \rho\, d^3x$, is *not* a Lorentz invariant quantity[1], due to its combination with the linear 3-momentum p^i into the 4-momentum $p^\mu = \left(E, p^i\right)$. We are forced therefore to look for a higher rank object encoding the relation among the energy density, the energy flux, the momentum density and the momentum flux or stress. This quantity is the so-called *energy-momentum-stress* tensor. Let's construct it.

[1]Note that in the electromagnetic case, the total electric charge $Q = \int \rho_e d^3x$ *is* Lorentz invariant.

6.1.1 Newtonian fluids

While point particles are characterized by their energy and momentum, the motion of continuous matter is usually characterized by two quantities: the mass density $\rho(t, x^i)$ and the velocity of the fluid $v(t, x^i)$, which generally depend on space and time. The evolution of a continuous system is determined by two equations:

i) A continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial \left(\rho v^j\right)}{\partial x^j} = 0\,, \tag{6.2}$$

reflecting the fact that mass is neither created or destroyed in Classical Mechanics (the flowing of mass out from a volume is equal to the loss of mass in it).

ii) A Newton's 2nd law for fluids

$$f^i = \rho a^i = \rho \left(\frac{\partial v^i}{\partial t} + v^j \frac{\partial v^i}{\partial x^j} \right)\,, \tag{6.3}$$

with

$$a^i = \lim_{\Delta t \to 0} \frac{v^i(t + \Delta t, x + \Delta x) - v^i(t, x)}{\Delta t}\,, \tag{6.4}$$

and $f^i = f^i(t, x)$ the total force per unit volume around a point x at time t. The so-called *total derivative* of the velocity field

$$\frac{Dv^i}{dt} \equiv \frac{\partial v^i}{\partial t} + v^j \frac{\partial v^i}{\partial x^j} \tag{6.5}$$

contains two pieces, the local derivative $\partial \mathbf{v}/\partial t$, which gives the change of the velocity $\mathbf{v}$ as a function of time at a given point in space, and the so-called *convective derivative*, $(\mathbf{v} \cdot \nabla)\,\mathbf{v}$, which represents the change of $\mathbf{v}$ for a moving fluid particle due to the inhomogeneity of the fluid vector field.

If we assume that there are not other forces apart from those exerted by the fluid on itself, we are left with internal forces like pressure or friction acting only between neighboring regions of matter. Consider a infinitesimal volume dV with surface area dA centered at a point x at time t. Let us denate by n_j the normal vector to the surface. In a perfect fluid[2], the force F^i exerted by the matter on the area is proportional to the area itself $F^i = p(t, x)\delta^{ij} n_j dA$, with $p(t, x)$ the pressure at that point at time t. In the most general case, we will also have *shear forces*

$$F^i(t, x) = T^{ij}(t, x) n_j dA\,, \tag{6.6}$$

due to the tendency of fluid elements moving with different velocities to drag adjacent matter. The coefficients T^{ij} are the components of the so-called *stress tensor*, which must be symmetric, $T^{ij} = T^{ji}$.

Exercise
Consider the 3-component of the torque acting on an infinitesimal cube of a material of density ρ and side length L. Compare it with the moment of inertia of the cube $I = \frac{1}{6}\rho L^5$. What happens if $T^{ij} \neq T^{ji}$ in the limit $L \to 0$?

[2] A perfect fluid is defined as one for which there are no forces between the particles, no heat conduction and no viscosity.

The total force exerted per unit area in a given direction[3] can be transformed into a total force by unit volume via the Gauss' theorem

$$-\int_A T^{ij} n_j dA = -\int_V \left(\partial_j T^{ij}\right) dV \qquad \longrightarrow \qquad f^i = -\frac{\partial T^{ij}}{\partial x^j} \,. \tag{6.7}$$

Plugging in this result into the Newton 2nd law (cf. Eq. (6.3))

$$\rho\left(\frac{\partial v^i}{\partial t} + v^j \frac{\partial v^i}{\partial x^j}\right) + \frac{\partial T^{ij}}{\partial x^j} = 0\,, \tag{6.8}$$

and using the continuity equation to write

$$\rho\frac{\partial v^i}{\partial t} = \frac{\partial\left(\rho v^i\right)}{\partial t} - v^i\frac{\partial \rho}{\partial t} = \frac{\partial\left(\rho v^i\right)}{\partial t} + v^i\frac{\partial\left(\rho v^j\right)}{\partial x^j} \tag{6.9}$$

The previous result and the continuity equation (6.2), the Newton's 2nd law (6.3) for this particular case ($f^i = -\partial_j T^{ij}$) can be written as

$$\frac{\partial\left(\rho v^i\right)}{\partial x^0} + \frac{\partial}{\partial x^j}\left(\rho v^i v^j + T^{ij}\right) = 0\,, \tag{6.10}$$

which is the so-called *Euler equation.*

6.1.2 Relativistic fluids

Eqs. (6.2) and (6.10) can be unified into a single equation in the framework of Special Relativity. To see this, note that the 3-velocity v^i is contained in the relativistic 4-velocity $u^\mu = \left(u^0, u^i\right) = \left(\gamma, \gamma v^i\right)$. Taking into account the non-relativistic limit of this relation, $u^\mu = \left(1, v^i\right)$, we can rewrite (6.2) and (6.10) as

$$\frac{\partial\left(\rho u^0 u^0\right)}{\partial x^0} + \frac{\partial\left(\rho u^0 u^j\right)}{\partial x^j} = 0\,, \qquad \frac{\partial\left(\rho u^i u^0\right)}{\partial x^0} + \frac{\partial}{\partial x^j}\left(\rho u^i u^j + T^{ij}\right) = 0\,, \tag{6.11}$$

which can be considered as parts of the single equation

$$\partial_\nu T^{\mu\nu} = 0\,, \qquad T^{\mu\nu} = \rho u^\mu u^\nu + t^{\mu\nu}\,, \tag{6.12}$$

with $t^{\mu\nu} = \text{diag}\left(0, T^{ij}\right)$. The quantity $T^{\mu\nu}$ is the so-called *energy-momentum-stress tensor* or in a shorter version the *energy-momentum tensor*[4] or the *stress-energy tensor.* It is a rank-2 symmetric tensor encoding all the information about energy density, momentum density, stress, pressure .... The ten components of this tensor have the following interpretation:

- T^{00} is the local energy density, including any potential contribution from forces between particles and their kinetic energy.
- T^{0i} is the energy flux in the i direction. This includes not only the bulk motion but also any other processes giving rise to transfers of energy, as for instance heat conduction.
- T^{i0} is the density of the momentum component in the i direction, i.e. the 3−momentum density. As the previous case, it also takes into account the changes in momentum associated to heat conduction.

[3] The minus sign appears because we are considering the force exerted on matter *inside* the volume by the matter *outside*

[4] This name can be sometimes misleading as it can be confused with the energy-momentum *4-vector* p^μ in sentences including things like "the energy-momentum conservation equation...". The difference should be always clear from the context.

- T^{ij} is the 3-momentum flux or stress tensor, i.e the rate of flow of the i momentum component per unit area in the plane orthogonal to the j-direction. The component T^{ii} encodes the isotropic pressure in the i direction while the components T^{ij} with $i \neq j$ refer to the *viscous stresses* of the fluid.

6.1.3 Relativistic perfect fluids

A relativistic perfect fluid is defined to be one in which the $t^{\mu\nu}$ part of the stress-energy tensor $T^{\mu\nu}$, as seen in a local reference frame moving along with the fluid, has same form as the non-relativistic perfect fluid

$$t^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} . \tag{6.13}$$

Heat conduction, viscosity or any other transport or dissipative processes in this case are negligible. The form of Eq. (6.13) in an arbitrary inertial frame can be obtained by performing a general Lorentz transformation

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & \gamma v^i \\ \gamma v^i & \delta^{ij} + v^i v^j \left(\gamma - 1\right)/v^2 \end{pmatrix} = \begin{pmatrix} u^0 & u^i \\ u^i & \delta^{ij} + u^i u^j/(1+\gamma) \end{pmatrix} , \tag{6.14}$$

moving from the rest frame $u^\mu = (1, \mathbf{0})$ to one in which the fluid moves with 3-velocity v^i. We get

$$\bar{t}^{\mu\nu} = \Lambda^{\nu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} t^{\rho\sigma} = p\left(\eta^{\mu\nu} + u^\mu u^\nu\right) , \tag{6.15}$$

with u^μ the 4-velocity vector field tangent to the worldines of the fluid particles. Taking into account this result, the full stress-energy tensor (6.12) takes the form

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p\eta^{\mu\nu} . \tag{6.16}$$

The resulting equation is manifestly covariant and can be easily generalized to arbitrary coordinate systems or curved spacetimes by simply replacing the local metric $\eta^{\mu\nu}$ by a general metric $g^{\mu\nu}$

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p g^{\mu\nu} . \tag{6.17}$$

The conservation law $\partial_\nu T^{\mu\nu} = 0$ in Eq. (6.12) becomes a *local conservation law*

$$\nabla_\nu T^{\mu\nu} = 0 \tag{6.18}$$

in which the standard derivative ∂_μ is replaced by the covariant derivative ∇_μ. The word *local* is, as always in this course, important. Eq. (6.18) is *not* a conservation law, nor should it be. As we will see, energy is not conserved in the presence of dynamical spacetime curvature but rather changes in response to it.

Exercise

Prove Eq. (6.15).

6.2 The microscopic description

The relation between ρ and p is usually characterized by an equation of state $p = p(\rho)$ which depends on the microscopic particles involved in the fluid. In order to get some insight about the possible equations of state, let me consider a macroscopic collection of N structureless point particles interacting through spatially localized collisions. The energy density associated to any of them is given by

$$T_n^{00} = E_n\, \delta^{(3)}\left(\mathbf{x} - \mathbf{x}_n(t)\right) = m_n \gamma_n \delta^{(3)}\left(\mathbf{x} - \mathbf{x}_n(t)\right)\,, \tag{6.19}$$

with $\gamma_n = 1/\sqrt{1-v_n^2}$ and $n = 1,\ldots,N$ a label selecting the particular particle we are referring to. Taking into account the identity

$$\begin{aligned} \int_{-\infty}^{+\infty} d\tau\, \delta^{(4)}\left(x - x(\tau)\right) &= \int_{-\infty}^{+\infty} d\tau\, \delta\left(t - t(\tau)\right) \delta^{(3)}\left(\mathbf{x} - \mathbf{x}(\tau)\right) \\ &= \frac{d\tau}{dt} \delta^{(3)}\left(\mathbf{x} - \mathbf{x}(t)\right) = \frac{1}{\gamma} \delta^{(3)}(\mathbf{x} - \mathbf{x}(t))\,, \end{aligned} \tag{6.20}$$

the non-Lorentz invariant 3-dimensional Dirac delta appearing in Eq. (6.23) can be transformed into a Lorentz invariant 4-dimensional Dirac delta[5]

$$T_n^{00} = m_n \int_{-\infty}^{+\infty} d\tau_n\, u_n^0 u_n^0 \delta^{(4)}\left(x - x_n(\tau_n)\right)\,. \tag{6.21}$$

The same procedure can be applied to the spatial momentum density (or energy current) of the particle

$$T_n^{0i} = p_n^i\, \delta^{(3)}\left(\mathbf{x} - \mathbf{x}_n(t)\right) = m_n \gamma_n v_n^i\, \delta^{(3)}\left(\mathbf{x} - \mathbf{x}_n(t)\right) = E_n v_n^i\, \delta^{(3)}\left(\mathbf{x} - \mathbf{x}_n(t)\right)\,, \tag{6.22}$$

and to the flux of the i momentum component in the j direction (or viceversa)

$$T_n^{ij} = p_n^i v_n^j\, \delta^{(3)}\left(\mathbf{x} - \mathbf{x}_n(t)\right) = p_n^j v_n^i\, \delta^{(3)}\left(\mathbf{x} - \mathbf{x}_n(t)\right)\,. \tag{6.23}$$

We obtain

$$T_n^{0i} = m_n \int_{-\infty}^{+\infty} d\tau_n\, u_n^0 u_n^i \delta^{(4)}\left(x - x_n(\tau_n)\right)\,, \qquad T_n^{ij} = m_n \int_{-\infty}^{+\infty} d\tau_n\, u_n^i u_n^j \delta^{(4)}\left(x - x_n(\tau_n)\right)\,. \tag{6.24}$$

Eqs. (6.21) and (6.24) can be rewritten in a very compact way in terms of the stress-energy-momentum tensor $T^{\mu\nu}$

$$T_n^{\mu\nu} = m_n \int_{-\infty}^{+\infty} d\tau_n\, u_n^\mu u_n^\nu \delta^{(4)}\left(x - x_n(\tau_n)\right) = \int_{-\infty}^{+\infty} d\tau_n\, \frac{p_n^\mu p_n^\nu}{m_n} \delta^{(4)}\left(x - x_n(\tau_n)\right)\,, \tag{6.25}$$

which is manifestly symmetric and Lorentz invariant since $u_n^\mu u_n^\nu$ is a tensor under Lorentz transformations and both m_n and $d\tau_n\, \delta^{(4)}\left(x - x_n(\tau_n)\right)$ are Lorentz scalars. The total energy density of the whole system of particles can be written as the sum of the individual contributions, namely

$$T^{\mu\nu} = \sum_{n=1}^{N} T_n^{\mu\nu}\,. \tag{6.26}$$

6.2.1 Energy-momentum tensor conservation and geodesics

Let us see under which conditions the total energy momentum tensor (6.26) is conserved. Taking the derivative with respect to the coordinates we get

$$\partial_\mu T^{\mu\nu} = \sum_{n=1}^{N} m_n \int_{-\infty}^{+\infty} d\tau_n\, u_n^\mu u_n^\nu \partial_\mu \delta^{(4)}\left(x - x_n(\tau_n)\right)\,, \tag{6.27}$$

[5] The fact that the 4-Dimensional Dirac delta $\delta^{(4)}(x)$ is Lorentz invariant follows directly from the definition $\int d^4x \delta^{(4)}(x) = 1$ and the fact that the volume element d^4x is Lorentz invariant.

which using

$$u_n^\mu \partial_\mu \delta^{(4)}\left(x - x_n(\tau_n)\right) = \frac{dx_n^\mu}{d\tau_n}\frac{\partial}{\partial x^\mu}\delta^{(4)}\left(x - x_n(\tau_n)\right) = -d/d\tau_n \delta^{(4)}\left(x - x_n(\tau_n)\right) , \tag{6.28}$$

can be written as

$$\partial_\mu T^{\mu\nu} = -\sum_{n=1}^{N} m_n \int_{-\infty}^{+\infty} d\tau_n \frac{d}{d\tau_n}\left(u_n^\nu \delta^{(4)}\left(x - x_n(\tau_n)\right)\right) + \sum_{n=1}^{N} m_n \int_{-\infty}^{+\infty} d\tau_n\, \dot{u}_n^\nu \delta^{(4)}\left(x - x_n(\tau_n)\right) .$$

The first term in the right hand side of the previous expression disappears in the particles are stable, i.e. if the orbits are closed or come from negative infinite time and disappear into positive infinite time. We are left then with the second term, which can be written as

$$\partial_\mu T^{\mu\nu} = \sum_{n=1}^{N} \int_{-\infty}^{+\infty} d\tau_n \frac{dp_n^\nu}{d\tau_n}\delta^{(4)}\left(x - x_n(\tau_n)\right) = \sum_{n=1}^{N} \frac{dp_n^\nu}{dt}\delta^{(3)}\left(\mathbf{x} - \mathbf{x}_n\right) , \tag{6.29}$$

with $p_n^\nu = m_n u_n^\nu$ the 4-momentum of the individual particles. The *local energy momentum conservation* $\partial_\mu T^{\mu\nu} = 0$ requires the particles to be free. Or in other words, the condition $\partial_\mu T^{\mu\nu} = 0$ is equivalent to the geodesic equation in Minkowski space-time, $dp_n^\mu/d\tau = 0$. This will be also the case in curved spacetime.

6.2.2 The fluid limit

On distances d much larger than the typical mean free path a, the number of particles is large and the statistical fluctuations about the mean properties of the fluid are expected to be small[6]. Imagine a comoving observer exploring distances $d \gg a$. If the fluid is isotropic[7], the average value of the $T^{0i} \propto u^0 u^i$ component measured by this observer will be zero since the vector u^i points in all possible directions. In this case, the fluid can be characterized in terms of two quantities: its mean density and pressure over the volume $\Delta V = d^3$

$$\rho = \left\langle \sum_n E_n \delta^{(3)}\left(\mathbf{x} - \mathbf{x}_n\right) \right\rangle_{\Delta V} , \qquad p = \frac{1}{3}\sum_i \left\langle \sum_n p_n^i v_n^i \delta^{(3)}\left(\mathbf{x} - \mathbf{x}_n\right) \right\rangle_{\Delta V} . \tag{6.30}$$

A simple inspection of Eqs. (6.30) reveals that, for standard matter, $0 \leq p \leq \rho/3$. In any other reference frame, the energy-momentum tensor for the perfect fluid reads

$$T^{\mu\nu}(x) = (\rho(x) + p(x))u^\mu(x)u^\nu(x) + p(x)\eta^{\mu\nu} , \tag{6.31}$$

with $u^\mu(x)$ denoting now the average value of the 4-velocities u_i^μ of the individual particles N_R inside the volume[8]. The perfect fluid form (6.31) can be used to model very different physical situations that often fall into one of the following categories:

1. **Non-relativistic matter**: For small velocities the dispersion relation $E_n = \sqrt{m_n^2 + \mathbf{p}_n^2}$ can be approximated by $E_n \simeq m_n + \mathbf{p}_n^2/2m_n$, which plugged back into (6.30) gives rise to $\rho \simeq m_n n + \frac{3}{2}p$. Taking into account that the statistical definition of temperature T is twice the energy possessed by each degree of freedom and assuming a monoatomic gas with 3 kinetic degrees of freedom, we can write $T = (2/3) \times \mathbf{p}_n^2/2m_n$ and therefore $\rho \simeq m_n n + \frac{3}{2}T$.

[6] Remember that, when we later apply the Equivalence Principle, we will have another scale into play: the scale L at which the gravitational effects start to be important. If this scale happens to be much larger than the scale d ($L \gg d \gg a$), the mean properties of the fluid can be safely considered as constant over the region.

[7] i.e if the fluid is *perfect*.

[8] Note that, when writing $u^\mu(x), \rho(x)$ and $p(x)$ we are explicitly taking into account that the averages can vary from one region to another.

2. **Dust**: A perfect fluid with zero pressure. $p = 0$, $t^{\mu\nu} = 0$, $T^{\mu\nu} = \rho \,\mathrm{diag}\,(1, 0, 0, 0)$.
3. **Radiation**: A perfect highly relativistic fluid. In this case $E_n \simeq |\mathbf{p}_n| \gg m_n$ and therefore[9] $\rho \simeq 3p$. The energy momentum for radiation is traceless, $T = T^{\mu}{}_{\mu} = \eta_{\mu\nu} T^{\mu\nu} = -\rho + 3p = 0$.

A worked-out example: The electromagnetic field

The paradigmatic case of a fluid with a radiation equation of state is the electromagnetic field. To see this explicitly, consider the energy density of the electromagnetic field

$$T_{00} = \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right) , \tag{6.32}$$

and write it in the way seen by an observer moving with 4-velocity u^{μ}. The electron field seen by that observer is given by

$$E_{\mu} = F_{\mu\nu} u^{\nu} . \tag{6.33}$$

Using this expression we get the following covariant expression for the square of the electric field

$$\mathbf{E}^2 = F_{\mu\nu} u^{\nu} F^{\mu}{}_{\rho} u^{\rho} . \tag{6.34}$$

A similar expression for the magnetic field square can be obtained from the explicit expression for the square of the electromagnetic field strength tensor

$$F_{\mu\nu} F^{\mu\nu} = -2\left(E^2 - B^2\right) . \tag{6.35}$$

We obtain

$$\mathbf{B}^2 = F_{\mu\nu} u^{\nu} F^{\mu}{}_{\rho} u^{\rho} + \frac{1}{2} F_{\mu\nu} F^{\mu\nu} . \tag{6.36}$$

Putting Eqs. (6.34) and (6.36) together, the covariant generalization of the energy density (6.32) becomes

$$\rho = \left(F_{\rho\mu} F^{\rho}{}_{\nu} - \frac{1}{4} F_{\rho\sigma} F^{\rho\sigma} \eta_{\mu\nu} \right) u^{\mu} u^{\nu} , \tag{6.37}$$

where we have inserted a factor $u_{\mu} u^{\mu} = -1$. The work is basically done. The quantity in parenthesis is the sought-for energy-momentum tensor for the electromagnetic field!

$$T_{\mu\nu} = F_{\rho\mu} F^{\rho}{}_{\nu} - \frac{1}{4} F_{\rho\sigma} F^{\rho\sigma} \eta_{\mu\nu} . \tag{6.38}$$

Exercise

- Compute the T^{0i} in terms of the electric and magnetic fields. Do you recognize the result?
- Prove that the electromagnetic energy-momentum is symmetric $T_{\mu\nu} = T_{\nu\mu}$ and traceless, $T^{\mu}{}_{\mu} = 0$. The electromagnetic field behaves as a fluid with equation of state $p = 1/3\rho$.

[9] Note that the quantity $\sum_i p_n^i v_n^i$ in Eq. (6.30) can be written as $\sum_i p_n^i v_n^i = \sum_i m \gamma_n v_n^i v_n^i = \frac{|\mathbf{p}_n|^2}{E_n}$, which goes to $|\mathbf{p}_n|$ when $E_n \simeq |\mathbf{p}_n|$.

6.3 Einstein equations: Heuristic derivation

We have finally all the tools needed to derive the Einstein field equations for the gravitational field. In the Poisson equation, the gravitational field is determined by the matter distribution. The relativistic version of the matter distribution[10], the energy-momentum tensor $T^{\rm M}_{\mu\nu}$, must be somehow equated[11] to some tensor $K_{\mu\nu}$ depending of the metric $g_{\mu\nu}$ and its first and second derivatives[12]

$$K_{\mu\nu} = \kappa^2 T^{\rm M}_{\mu\nu}\,, \tag{6.39}$$

with κ^2 a proportionality constant to be determined. But, what tensor? Einstein got the answer to this question trough a complicated process of intuition, trial and error; *superhuman exertions* in his own words. As claimed above, the left-hand side of Eq.(6.39) should contain a second order differential operator acting on the metric. We already found some quantities with this property in the previous chapter: the Riemann tensor $R^\mu{}_{\nu\rho\sigma}$ and its contractions. The most natural tentative for $K_{\mu\nu}$ would be the Ricci tensor $R_{\mu\nu}$, since this is the contraction appearing in the Newtonian limit of the geodesic deviation equation ($R^i{}_{0i0} = E^i{}_i$). This was also one of the first *trial and error* choices of Einstein

$$R_{\mu\nu} \approx \kappa^2 T^{\rm M}_{\mu\nu}\,. \tag{6.40}$$

Note however that this choice is inconsistent, since the divergence $\nabla^\nu R_{\mu\nu}$ of the Ricci tensor is, in general, different from zero and, according to our minimal coupling prescription, the energy-momentum should be locally conserved, $\nabla^\nu T^{\rm M}_{\mu\nu} = 0$. Indeed, making use of the Bianchi identity (5.71) we can write $\nabla^\mu R_{\mu\nu} = 1/2\nabla_\nu R$, which together with the trace of Eq. (6.40), $R = \kappa^2 g^{\mu\nu} T^{\rm M}_{\mu\nu} = \kappa^2 T^{\rm M}$, implies the condition $\nabla_\mu T^{\rm M} = 0$. Since the covariant derivative of the scalar quantity $T^{\rm M}$ is just the partial derivative, we should necessarily have a constant $T^{\rm M}$ throughout the whole spacetime, which is highly implausible, since, as we know, $T^{\rm M} = 0$ for the electromagnetic field and $T^{\rm M} > 0$ for standard matter. On top of that, Eq. (6.40) hides 10 differential equations for 6 physical unknowns: the components of the metric that cannot be freely changed by performing coordinates transformations in the 4 coordinates. We have to try harder.

The most general combination of symmetric tensors involving up to two derivatives of the metric is

$$K_{\mu\nu} = R_{\mu\nu} + a g_{\mu\nu} R + \Lambda g_{\mu\nu} \tag{6.41}$$

with a and Λ some unknown constants to be determined[13]. Imposing the local conservation of the energy-momentum tensor $\nabla^\mu T^{\rm M}_{\mu\nu}{=}0$ in Eq. (6.39) we get

$$\nabla^\mu K_{\mu\nu} = \nabla^\mu \left(R_{\mu\nu} + a g_{\mu\nu} R\right) = 0\,, \tag{6.42}$$

where we have taken into account that the covariant derivative ∇_μ is metric compatible and therefore $\nabla_\mu\left(\Lambda g^{\mu\nu}\right) = 0$. Our situation now is much better than that of Einstein, we are aware of the contracted form of the Bianchi identities[14] (5.71) and know the precise value of a that satisfies Eq. (6.42), namely $a = 1/2$. Taking this into account, we can rewrite Eq. (6.39) as

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa^2 T^{\rm M}_{\mu\nu}\,, \tag{6.43}$$

with

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\,, \tag{6.44}$$

[10] *Matter* should be understood in a broad sense, meaning really matter, radiation etc...

[11] A relativistic generalization should take the form of an equation between tensors.

[12] The requirement of having derivatives only up to second order is certainly reasonable. If this were not the case, one would have to specify for the *Cauchy problem* not only the value of the metric and its first derivative, but also higher derivatives on a spacelike surface.

[13] A possible proportionality constant in front of $R_{\mu\nu}$ has been factored out and incorporated in the still unknown factors κ and *Lambda* in the right hand side of Eq. (6.39).

[14] He wasn't.

the Einstein tensor defined in previous chapter (cf. Eq. (5.72)) and Λ the famous *cosmological constant* term. Writing this cosmological constant term in the right hand side of the equation, we can interpret it as the energy-momentum tensor of a fluid with a weird equation of state $p = -\rho$

$$G_{\mu\nu} = \kappa^2 \left(T^{\mathrm{M}}_{\mu\nu} + T^{\Lambda}_{\mu\nu}\right) , \qquad T^{\Lambda}_{\mu\nu} = -\frac{\Lambda}{\kappa^2} g_{\mu\nu} \, . \tag{6.45}$$

Defining $T_{\mu\nu} \equiv T^{\mathrm{M}}_{\mu\nu} + T^{\Lambda}_{\mu\nu}$, we can write

$$G_{\mu\nu} = \kappa^2 T_{\mu\nu} \, . \tag{6.46}$$

Even though our derivation was quite heuristic, the solution that we have obtained is unique (Lovelock theorem). The resulting tensorial equation is a set of ten differential equations[15] for the metric $g_{\mu\nu}(x)$ given the energy-momentum tensor $T_{\mu\nu}(x)$. However, due to the existence of the Bianchi identities, not all the components are longer independent. There are only 6 independent equations to determine 6 independent components of the metric tensor.

As differential equations they are very complicated, even in vacuum. Both the Ricci scalar and the scalar curvature involve derivatives and products of Christoffel symbols, which in turn involve derivatives of the metric tensor. There is also some dependence on the metric hidden in the energy-momentum tensor. On top of that, the equations are not linear, as it should be expected, since, according to the Equivalence Principle, every form of energy, including the *gravitational self-energy*, must be a source of the gravitational field[16]. The non-linearity of the equation forbids us to apply the superposition principle, given two known solutions they cannot be combined to get a new one.

The Einstein equation in words

The physical meaning of Einstein equations can be clarified by considering an observer with velocity u^μ. The energy density as measured in the energy frame of such an observer is given by $\rho = T_{\mu\nu} u^\mu u^\nu$. Taking this into account, together the interpretation of the Einstein tensor that we developed in the previous Chapter, the physical content of (6.43) can be summarized as

$$\left(G_{\mu\nu} - \kappa^2 T_{\mu\nu}\right) u^\mu u^\nu = 0 \, , \tag{6.47}$$

which in words reads

$$\left[\begin{array}{c}\text{Scalar curvature of the spatial}\\ \text{sections measured by an}\\ \text{observer with velocity } u^\mu\end{array}\right] = 2\kappa^2 \left[\begin{array}{c}\text{Energy density measured by}\\ \text{an observer with 4-velocity } u^\mu\end{array}\right] .$$

[15] Both sides of the equation are symmetric rank-2 tensors.

[16] Note however that they have a well-posed initial-value structure, i.e. they determine the future values of $g_{\mu\nu}$ from given initial data. This consideration is of key importance for the study of systems evolving in time from some initial state, as for instance, gravitational waves.

Newton	Einstein
Newton 2nd law $\frac{d^2x^i}{dt^2} = -\delta^{ij}\frac{\partial\Phi}{\partial x^j}$	Geodesic equation $\frac{d^2x^\mu}{d^2\sigma} = -\Gamma^\mu{}_{\nu\rho}\frac{dx^\rho}{d\sigma}\frac{dx^\nu}{d\sigma}$
Tidal deviation $\frac{d^2\xi^i}{dt^2} = -E^i{}_j\,\xi^j$	Geodesic deviation $\frac{D^2\xi^\mu}{d\sigma^2} = -R^\mu{}_{\nu\rho\sigma}u^\nu u^\sigma \xi^\rho$
1st Bianchi identity $E_{ij} = E_{ji}$	1st Bianchi identitiy $R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} = 0$
2nd Bianchi identity $E^i{}_{[j,l]} = 0$	2nd Bianchi identitiy $\nabla_\kappa R^\mu{}_{\nu\rho\sigma} + \nabla_\sigma R^\mu{}_{\nu\kappa\rho} + \nabla_\rho R^\mu{}_{\nu\sigma\kappa} = 0$
mass density ρ	Energy-momentum tensor $T_{\mu\nu}$
Poisson equation $E^i{}_i = 4\pi G\rho$	Einstein equation $G_{\mu\nu} = 8\pi G T_{\mu\nu}$
single elliptic equation	10 coupled equations 4 elliptic and 6 hyperbolic
boundary data required	initial and boundary data required

Table 6.1: Newtonian vs Einstenian description of gravity.

6.4 The linearized theory of gravity

Equation (6.46) looks very promising but we have still to prove that it is able to reproduce the Newtonian theory of gravity and determine the value of the unknown constants κ and Λ. The fastest way to obtain the Newtonian limit is to use the assumptions discussed in Section 3.6. Let me however relax these assumptions and obtain the general expression for the Einstein equation in the so-called *weak field limit*. This limit is defined by the condition

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \qquad \text{with} \qquad |h_{\mu\nu}| \ll 1\,. \tag{6.48}$$

The quantity $h_{\mu\nu}$ is then understood as a small perturbation on top of the Minkowski background. Consistently with this point of view, we will raise and lower its indices with the flat Minkowski metric $\eta_{\mu\nu}$, namely $h^\mu{}_\sigma = \eta^{\mu\rho}h_{\rho\sigma}\,, h^{\mu\nu} = \eta^{\nu\sigma}h^\mu{}_\sigma\,.$

In order to compute the expression for the Einstein tensor $G_{\mu\nu}$ at the lowest order in perturbation theory we must first determine the linearized version of the Ricci tensor and the scalar curvature, which are functions of the metric connection $\Gamma^{\mu}{}_{\nu\rho}$. Inserting the expansion (6.48) into the definition of the metric connection, we get

$$\Gamma^{\mu}{}_{\nu\rho} = \frac{1}{2}\eta^{\mu\sigma}\left(\partial_\nu h_{\sigma\rho} + \partial_\rho h_{\sigma\nu} - \partial_\sigma h_{\rho\nu}\right) + \mathcal{O}(h^2_{\mu\nu})\,. \tag{6.49}$$

The next step is to compute the 4 pieces of Riemman tensor, which, written in a very schematic way, have the structure $R^{\cdot\cdot\cdot}_{\cdot\cdot\cdot} \sim \partial\Gamma - \partial\Gamma + \Gamma\Gamma + \Gamma\Gamma$. Taking into account (6.49), we realize that only the first two terms ($\sim \partial\Gamma$) give a contribution to the leading order

$$R^{\mu}{}_{\nu\rho\sigma} = \frac{1}{2}\partial_\rho\left(\partial_\sigma h^{\mu}{}_{\nu} + \partial_\nu h^{\mu}{}_{\sigma} - \partial^{\mu} h_{\nu\sigma}\right) - (\rho \leftrightarrow \sigma) = \frac{1}{2}\left(\partial_\nu\partial_\rho h^{\mu}{}_{\sigma} + \partial_\sigma\partial^{\mu} h_{\nu\rho} - (\rho \leftrightarrow \sigma)\right)\,. \tag{6.50}$$

The linearized version of the Ricci tensor and the scalar of curvature can be computed by simply performing contractions in the previous expression. Denoting respectively by $h \equiv h^{\mu}{}_{\mu}$ and $\Box = \partial^\mu\partial_\mu$ the trace of the perturbation tensor and the d'Alambertian operator and contracting the indices μ and σ in Eq. (7.17), we get[17]

$$R_{\nu\rho} = -\frac{1}{2}\left(\Box h_{\nu\rho} + \partial_\nu\partial_\rho h - \partial_\nu\partial_\sigma h^{\sigma}{}_{\rho} - \partial_\rho\partial_\sigma h^{\sigma}{}_{\nu}\right)\,, \tag{6.51}$$

which can be further contracted in the indices ν and ρ to obtain

$$R = R^{\nu}{}_{\nu} = \eta^{\nu\rho}R_{\nu\rho} = -\Box h + \partial_\nu\partial_\sigma h^{\nu\sigma}\,. \tag{6.52}$$

Collecting all the terms and inserting them into the definition of the Einstein tensor (6.44), we get

$$\begin{aligned} G_{\nu\rho} &= -\frac{1}{2}\left(\partial_\nu\partial_\rho h + \Box h_{\nu\rho} - \partial_\nu\partial_\sigma h^{\sigma}{}_{\rho} - \partial_\rho\partial_\sigma h^{\sigma}{}_{\nu} - \eta_{\nu\rho}\Box h + \eta_{\nu\rho}\partial_\mu\partial_\sigma h^{\mu\sigma}\right) \\ &= -\frac{1}{2}\left(\Box\tilde{h}_{\nu\rho} + \eta_{\nu\rho}\partial_\mu\partial_\sigma\tilde{h}^{\mu\sigma} - \partial_\nu\partial_\sigma\tilde{h}^{\sigma}{}_{\rho} - \partial_\rho\partial_\sigma\tilde{h}^{\sigma}{}_{\nu}\right)\,, \end{aligned} \tag{6.53}$$

where in the last step we have defined the so-called *trace reverse*

$$\tilde{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h\,, \qquad\qquad h_{\mu\nu} = \tilde{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\tilde{h}\,, \tag{6.54}$$

which keeps track of the extra terms obtained when passing from $R_{\nu\rho}$ to $G_{\nu\rho}$. The name trace reverse comes from the property $\tilde{h} \equiv \tilde{h}^{\mu}{}_{\mu} = -h$. Note also the useful properties

$$\tilde{\tilde{h}}_{\mu\nu} = h_{\mu\nu}\,, \qquad G_{\mu\nu} = \tilde{R}_{\mu\nu}\,. \tag{6.55}$$

The linearized Einstein equations becomes finally

$$\left(\Box\tilde{h}_{\nu\rho} + \eta_{\nu\rho}\partial_\mu\partial_\sigma\tilde{h}^{\mu\sigma} - \partial_\nu\partial_\sigma\tilde{h}^{\sigma}{}_{\rho} - \partial_\rho\partial_\sigma\tilde{h}^{\sigma}{}_{\nu}\right) = -2\kappa^2 T_{\nu\rho}\,. \tag{6.56}$$

The resulting expression is rather involved, but fortunately we still have some freedom to play with: the gauge freedom.

[17] The global minus sign comes from the permutation of the last two indices two construct the Ricci scalar.

Gauge fixing

Eqs. (7.17) and (6.53), and therefore (6.56), are invariant under the transformation

$$h_{\nu\rho} \longrightarrow h_{\nu\rho} - \partial_\nu \xi_\rho - \partial_\rho \xi_\nu \,, \tag{6.57}$$

as can be easily verified be performing the explicit computation. This kind of change is called a *gauge transformation*, due to the strong analogy with the *gauge transformations* in the electromagnetic theory. The simplest way to understand this *gauge freedom* is to trace it back to the transformation of the full metric $g_{\mu\nu}$. Consider an infinitesimal transformation $x^\mu \to \bar{x}^\mu = x^\mu + \xi^\mu$. Under such a transformation the metric changes to

$$\begin{aligned} \bar{g}^{\mu\nu}\left(x^\rho + \xi^\rho\right) &= g^{\rho\sigma}(x^\rho)\frac{\partial \bar{x}^\mu}{\partial x^\rho}\frac{\partial \bar{x}^\nu}{\partial x^\sigma} \\ &= g^{\rho\sigma}\left(\delta^\mu{}_\rho + \partial_\rho \xi^\mu\right)\left(\delta^\nu{}_\sigma + \partial_\sigma \xi^\nu\right) \\ &= g^{\mu\nu}(x^\rho) + g^{\mu\sigma}\partial_\sigma \xi^\nu + g^{\nu\rho}\partial_\rho \xi^\mu \,. \end{aligned} \tag{6.58}$$

Expanding the left-hand side of this equation in a Taylor series in ξ^ρ and retaining only the terms up to linear order, we get

$$\bar{g}^{\mu\nu}(x^\rho) = g^{\mu\nu}(x^\rho) + \delta g^{\mu\nu} \,, \tag{6.59}$$

with

$$\delta g^{\mu\nu} \equiv -\xi^\rho \partial_\rho g^{\mu\nu} + g^{\mu\rho}\partial_\rho \xi^\nu + g^{\nu\rho}\partial_\rho \xi^\mu = \nabla^\nu \xi^\mu + \nabla^\mu \xi^\nu \,. \tag{6.60}$$

In the particular case in which the perturbation is performed around the Minkowski background, $g_{\mu\nu} = h_{\mu\nu} + \eta_{\mu\nu}$, the covariant derivatives in (6.59) become standard derivatives and we recover the transformation law (6.57). The linearized theory is invariant under (6.57) because the full nonlinear theory is invariant under general coordinate transformations! This is extremely interesting, since it allows us to further simplify the linearized version of the Einstein tensor by simply performing infinitesimal coordinates transformations, or in other words, changes from a splitting $g_{\mu\nu} = \eta_{\mu\nu} + h_{\rm old}$ to a different splitting $g_{\mu\nu} = \eta_{\mu\nu} + h_{\rm new}$. A simple inspection of Eq. (6.56) reveals that an interesting condition to be satisfied by the trace reverse tensor in the new coordinate system would be the tensor analog of the Lorenz gauge $\partial_\mu A^\mu = 0$ in the electromagnetic theory[a], namely

$$\partial_\rho \tilde{h}^{\nu\rho}_{\rm new} = 0 \,. \tag{6.61}$$

Let us see if we are allowed to choose such a gauge. The change in the trace reverse tensor $\tilde{h}_{\mu\nu}$ follows directly from Eqs. (6.54) and (6.57)

$$\tilde{h}^{\nu\rho}_{\rm new} = \tilde{h}^{\nu\rho}_{\rm old} - \partial^\nu \xi^\rho - \partial^\rho \xi^\nu + \eta^{\nu\rho}\partial_\mu \xi^\mu \,. \tag{6.62}$$

Taking the derivative of this equation we get

$$\partial_\rho \tilde{h}^{\nu\rho}_{\rm new} = \partial_\rho \tilde{h}^{\nu\rho}_{\rm old} - \Box \xi^\nu \,. \tag{6.63}$$

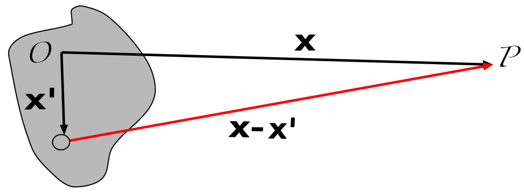
In order to satisfy the gauge fixing (6.61), ξ^ν must be a solution of the inhomogeneous wave equation

$$\Box \xi^\nu = \partial_\rho \tilde{h}^{\nu\rho}_{\rm old} \,. \tag{6.64}$$

The existence of a solution transforming from an arbitrary $h_{\mu\nu}$ to the so-called *Lorenz gauge* $\partial_\rho \tilde{h}^{\nu\rho}_{\rm new} = 0$ is guaranteed[b] for sufficiently well behaved $\partial_\rho \tilde{h}^{\nu\rho}_{\rm old}$. In fact, the choice is not unique since we can always add to it any solution of the homogeneous wave equation $\Box \xi^\nu_H = 0$ and the result will still obey $\Box\left(\xi^\nu + \xi^\nu_H\right) = \partial_\rho \tilde{h}^{\nu\rho}_{\rm old}$. The Lorenz gauge $\partial_\rho \tilde{h}^{\nu\rho}_{\rm new} = 0$ is actually a set of gauges.

[a] It "kills" three of the four terms in (6.53).

[b] As you learnt in your electrodynamic course, the solution of this equation can be obtained by means of the retarded Green functions of the d'Alambertian operator.



In view of the previous discussion, we realize that most of the terms in the left-hand side of Eq. (6.56) merely serve to maintain gauge invariance. When the *Hilbert gauge condition*[18] $\partial_\rho \tilde{h}^{\nu\rho} = 0$ is imposed, the linearized version of the Einstein equation simplifies dramatically

$$\Box \tilde{h}_{\mu\nu} = -2\kappa^2 T_{\mu\nu} \,. \tag{6.65}$$

This equation is formally identical to the Maxwell equations in the Lorenz gauge and can be solved by using the Green's function method.

Green's functions

Consider a differential wave equation of the form

$$\Box f(t, \mathbf{x}) = s(t, \mathbf{x}) \,, \tag{6.66}$$

with $f(t, \mathbf{x})$ a radiation field and $s(t, \mathbf{x})$ a source term. A Green's function $G(t, \mathbf{x}; t', \mathbf{x}')$ is defined as the field generated at the point $(t, \mathbf{x})$ by a delta function source at $(t', \mathbf{x}')$. i.e.

$$\Box G(t, \mathbf{x}; t', \mathbf{x}') = \delta(t - t')\delta(\mathbf{x} - \mathbf{x}') \,. \tag{6.67}$$

The field due the actual source $s(t, \mathbf{x})$ can be obtained by integrating the Green's function against $s(t, \mathbf{x})$:

$$f(t, \mathbf{x}) = \int dt' d^3x' \, G(t, \mathbf{x}; t', \mathbf{x}') \, s(t', \mathbf{x}') \quad . \tag{6.68}$$

Physically the Green's function approach merely reflects the fact that (6.66) is a linear equation. The full solution of the equation can be obtained by solving for a point source and adding the resulting waves from each point inside the source.

The Green's function associated with the wave operator $\Box$ is very well known (see for instance the Jackson's book on electrodynamics.):

$$G(t, \mathbf{x}; t', \mathbf{x}') = -\frac{\delta(t' - [t - |\mathbf{x} - \mathbf{x}'|])}{4\pi|\mathbf{x} - \mathbf{x}'|} \,. \tag{6.69}$$

Exercise

Derive this equation in case you haven't done it before.

Using (6.69) into (6.65), we get[19]

$$\tilde{h}_{\mu\nu} = \frac{\kappa^2}{2\pi} \int \frac{T_{\mu\nu}\left(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}'\right)}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}' \,, \tag{6.70}$$

[18] This gauge is also called Einstein gauge, harmonic gauge, de Donder gauge, Fock gauge or, in analogy with electromagnetism, Lorenz gauge.

[19] Note that we can always add to this particular solution an arbitrary solution of the homogeneous wave equation (vacuum). As in electromagnetism, the metric perturbation consists of the field generated by the source plus wave-like vacuum solutions propagating at the speed of light.

which is analogous to the relation between the vector potential A_μ and the current J_μ in electromagnetism. Note the argument $t - |\mathbf{x} - \mathbf{x}'| = t - |\mathbf{x} - \mathbf{x}'|/c$. Eq. (6.70) is a *retarded solution*[20], taking into account the lag associated with the propagation of information from events at $\mathbf{x}$ to position $\mathbf{x}'$. Gravitational influences propagate at the finite speed of light. Action at a distance is gone forever! We will be back to this point at the next chapter , but before let me finish our main task: determining the value of the constants κ^2 and Λ. For doing that let me consider the case we know better: the gravitational field created by a static spherical mass distribution of total mass M. The energy-momentum tensor for such a system has only one non-vanishing component (cf. Eq. (6.45))

$$T^{00} = \left(\rho + \frac{\Lambda}{\kappa^2}\right) \text{diag}\,(1,0,0,0)\ . \tag{6.72}$$

Plugging this into the time independent version of Eq. (6.70), we get

$$\tilde{h}_{00} = \frac{\kappa^2}{2\pi}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3\mathbf{x}' + \frac{1}{2\pi}\int\frac{\Lambda}{|\mathbf{x}-\mathbf{x}'|}d^3\mathbf{x}'\,, \qquad \tilde{h}_{0i} = 0\,, \qquad \tilde{h}_{ij} = 0\,. \tag{6.73}$$

If the mass distribution is concentrated around the origin ($\mathbf{x}' = 0$), the component h_{00} evaluated at a distance $r = |\mathbf{x} - \mathbf{x}'|$ becomes[21]

$$\tilde{h}_{00} = \frac{\kappa^2}{2\pi}\int \frac{\rho(\mathbf{x}')}{r}d^3\mathbf{x}' + \frac{1}{2\pi}\int\frac{\Lambda}{r}d^3\mathbf{x}' = \frac{\kappa^2}{2\pi}\frac{M}{r} + \frac{2}{3}\Lambda r^2 \tag{6.74}$$

with

$$M = \int \rho(\mathbf{x}')\,d^3\mathbf{x}' \tag{6.75}$$

the total mass of our spherical distribution. Taking now into account that $\tilde{h} = \eta^{\mu\nu}\tilde{h}_{\mu\nu} = -\tilde{h}_{00}$ and using the definition (6.54) we get

$$h_{00} = h_{11} = h_{22} = h_{33} = \frac{\kappa^2 M}{4\pi r} + \frac{1}{3}\Lambda r^2\,. \tag{6.76}$$

Comparing this result with that obtained by performing the weak field limit of the geodesic equation in the $\Lambda = 0$ case, $h_{00}^{\Lambda=0} = -2\Phi = 2GM/r$, allows us to identify the sought-for proportionality constant

$$\kappa^2 = 8\pi G\,. \tag{6.77}$$

When $\Lambda \neq 0$, the Newtonian potential becomes modified at long distances

$$\Phi = -\frac{GM}{r} - \frac{\Lambda}{6}r^2 \tag{6.78}$$

and line element takes the form

$$ds^2 = -\left(1 - \frac{2GM}{r} - \frac{1}{3}\Lambda r^2\right)dt^2 + \left(1 + \frac{2GM}{r} + \frac{1}{3}\Lambda r^2\right)dX^2\,, \tag{6.79}$$

with $dX^2 \equiv dx^2 + dy^2 + dz^2$. In Newtonian terms, a positive cosmological constant ($\Lambda > 0$) gives rise to a repulsive force per unit mass whose strength increases linearly with the distance

$$\mathbf{f} = -\frac{GM}{r^2}\,\mathbf{u}_r + \frac{\Lambda}{3}r\,\mathbf{u}_r\,, \tag{6.80}$$

[20] The retarded solution is obtained by imposing the Kirchoff-Sommerfeld "no-incoming radiation" boundary condition at past null infinity

$$\lim_{t\to\infty}(\partial_r + \partial_t)(r\bar{h}_{\mu\nu}) = 0\,, \tag{6.71}$$

with the limit taken along any surface with $ct + r$ =constant, together with the condition that $r\tilde{h}_{\mu\nu}$ and $r\partial_\rho\tilde{h}_{\mu\nu}$ are bounded in this limit.

[21] Note that the integral in over the prime variables!

Cosmological constant

If $\Lambda \neq 0$, it must be at least very small, $\rho^{\Lambda} \ll \rho^{\text{matter}}$, to avoid any observational effect in those situations in which the Newton's theory of gravity successfully explains the observations. Taking into account, for instance, that we do not see any modification of the Newtonian theory of gravity within the solar system, we can set the limit

$$|\rho_\Lambda| = \frac{|\Lambda|}{8\pi G} \leq \rho_{\text{Solar}} \qquad \longrightarrow \qquad |\rho_\Lambda| \leq \frac{3M_\odot}{4\pi R^3_{\text{Pluto}}} \simeq 10^{-29}\,\text{GeV}^4 \tag{6.81}$$

which, as assumed, makes the contribution of Λ completely negligible on the scale of the systems we will be interested in in this course[a] .

[a] It will play however a fundamental role at larger scales, as those you will considered in your Cosmology course.

	Linearized Gravity	Electromagnetism
Field equation	Einstein equation with $g_{\mu\nu} = h_{\mu\nu} + h_{\mu\nu}$	Maxwell equations
Basic potentials	Linearized metric $h_{\mu\nu}(x)$	4-vector potential $A^\mu = (\Phi, \mathbf{A})$
Sources	Energy-momentum tensor $T^{\mu\nu}$	4-vector current $J^\mu = (\rho, \mathbf{J})$
Lorenz gauge	$\partial_\mu \tilde{h}^{\mu\nu} = 0$ $\tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\text{h}$	$\partial_\mu A^\mu = 0$
Sourced wave equation	$\Box \tilde{h}_{\mu\nu} = -16\pi G T_{\mu\nu}$	$\Box A_\mu = J_\mu$
Solution	$\tilde{h}_{\mu\nu} = 4G \int \frac{[T_{\mu\nu}]_{\text{ret}}}{\lvert\mathbf{x}-\mathbf{x}'\rvert} d^3\mathbf{x}'$	$\tilde{A}_\mu = \frac{1}{4\pi} \int \frac{[J_\mu]_{\text{ret}}}{\lvert\mathbf{x}-\mathbf{x}'\rvert} d^3\mathbf{x}'$

Table 6.2: Linearized Einstein equations vs Maxwell equations.