CHAPTER 1

EUCLIDEAN SPACETIME AND NEWTONIAN PHYSICS

Absolute, true, and mathematical time, of itself, and from its own nature, flows equably without relation to anything external . . .

ISAAC NEWTON
Scholium of the Principia

The purpose of this chapter is to remind you the basic features of the Galilean spacetime and its symmetries, which are closely related to the form taken by Newton’s laws as seen by inertial observers. Although ideas presented in this chapter will be all familiar to you, the way of looking at them will be probably new. We will introduce some tensorial notation that will be useful in the future. Indeed, local differential geometry can be understood as a refinement of the tensorial methods presented here.

1.1 Galilean Relativity

Newtonian mechanics is based in two basic axioms:

1. Principle of Relativity: The laws of physics are the same in all the inertial frames: No experiment can measure the absolute velocity of an observer; the results of any experiment do not depend on the speed of the observer relative to other observers not involved in the experiment.

2. There exists an absolute time, which is the same for any observer.

1.2 Euclidean spacetime: old wine in a new bottle

When formulating mechanics in an axiomatic form, Newton, based on everyday experience, assumed the spacetime to be Euclidean $\mathbb{E}^1 \times \mathbb{E}^3$, i.e. an intrinsically flat and orientable metric space with trivial topology and well-defined distances and angles. A physical process in this spacetime (such as the collision of two particles) is called an event and it is independent of the particular choice of

\[1\] This is, at velocities much smaller than the velocity of light.
coordinates used for its description. The spatial location of the event can be specified in Cartesian coordinates \((x, y, z)\), in spherical coordinates \((r, \theta, \phi)\), or making use of any 3 independent numbers obtained by a well-defined coordinate transformation. However, among all the coordinates systems that can be used in Newtonian physics, the inertial coordinate systems are privileged (and least for Newton and Galileo). An inertial frame is a frame moving freely in spacetime, free of any force, which carries ideal clocks and measuring rods forming an orthonormal Cartesian coordinate system. In such a frame, a particular event \(P\) is characterized by 4 coordinates: its position \(\{x^i\} = \{x, y, z\} = \{x^1, x^2, x^3\}\),

\[
\{x^i\} = \{x, y, z\} = \{x^1, x^2, x^3\},
\]

and the time \(t\) at which it happens.

Time

Physical time is absolute (up to affine changes, see below) and it is used to characterize particle trajectories \(x^i(t)\). The temporal separation \(dt\) between two events is well-defined, independently of their spatial separation (see below). Simultaneous events are characterized by equal time surfaces separating the future and the past of the events. Any event may cause any simultaneous or later event.

Space

For each spatial coordinate we define a set of orthonormal basis vectors along the \(x^i\) coordinate direction

\[
\mathbf{e}_i = \{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}, \quad \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij},
\]

with \(\delta_{ij}\) the 3 dimensional Kronecker delta

\[
\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \text{diag}(1, 1, 1).
\]

We emphasize here that we do not consider \(\{x^i\}\) to be a vector since the homogeneity of space makes the choice of an origin completely arbitrary. The distance between points is the only significant quantity. On top of that, coordinates will no longer behave as a vector in the presence of gravity.
The infinitesimal displacement vector $d\mathbf{X}$ between two points (as any other vector in $\mathbb{E}^3$) can be expanded in terms of the basis vectors $\mathbf{e}_i$ as

$$d\mathbf{X} = \sum_{i=1}^{3} dx^i \mathbf{e}_i = dx^1 \mathbf{e}_1 + dx^2 \mathbf{e}_2 + dx^3 \mathbf{e}_3,$$

with $dx^i$ the so-called **contravariant components of the vector** in that orthonormal basis.

**Einstein summation convention**

Note the way in which we have located the indices in the previous equation. From now on, an index appearing twice in a product (in a superscript-subscript combination) will be understood to be automatically summed on or *contracted*. A quantity with no free tensor indices is said to be **fully contracted**. The name of the pair of contracted indices (latin indices $(i,j,k,\ldots)$ for the spatial coordinates or greek ones $(\mu,\nu,\rho,\ldots)$ in $3+1$ spacetime dimensions) is completely arbitrary and can be changed at will. For this reason, these indices are called **dummy indices**. Expressions with more than two repeated indices should never occur, being necessary in some cases to relabel them in order to avoid ambiguities. Non repeated indices are called **free indices** and must appear at the same level at both sides of the equations, for each independent term. As you will see, these rules are very useful, since they will allow us to reconstruct equations without any memorization, just by properly setting the indices up or down in the equation. On top of that, we will save a lot of time when writing expressions in General Relativity, which typically contain lots of indices. Using this convention, Eqs. (1.4) can be written as

$$d\mathbf{X} = \sum_{i=1}^{3} dx^i \mathbf{e}_i \quad \rightarrow \quad d\mathbf{X} = dx^1 \mathbf{e}_1.$$

**Exercise**

Which of the following expressions do not make sense or are ambiguous according to the previous rules? Why? Restore the sums on dummy indices in the rest of equations.

$$x^i = A_{ij}B^j x^k, \quad x^i = A^j_k B^k x^j, \quad D^{ij} = A^{ij}_k B^k C^j, \quad D^{ij} = A^{ik}_l B^k C^l,$$

$$x^i = A_{ij} x^j + B^k x^k, \quad x^i = A^j i x^j + B^j x^j, \quad D^{ij} = A^i_k B^k C^l.$$

The orthonormality of the basis vectors allows us to compute the contravariant components $dx^i$ as the scalar product of the vector $d\mathbf{X}$ and the corresponding basis vector $\mathbf{e}_i$

$$d\mathbf{X} \cdot \mathbf{e}_i = (dx^i \mathbf{e}_j) \cdot \mathbf{e}_i = dx^j (\mathbf{e}_j \cdot \mathbf{e}_i) = dx^j \delta_{ji} \equiv dx_i,$$

where in the last step we have defined the so-called **covariant components** $dx_i$

$$dx_i \equiv \delta_{ij} dx^j.$$

The 3 dimensional Kronecker delta $\delta_{ij}$ allows therefore to lower (or raise) spatial indices. The definition of covariant vectors is done only for notational brevity, there is nothing deep on it. The location of the indices in Euclidean space is just a clever way of keeping into account the summation convention and does not give rise to any change in the numerical value of the different components

$$dx^i = +dx_i.$$
As we will see in the next chapters, this is not the general case in a non-Cartesian reference frame or in other spacetimes with undefined metric, such as the Minkowski spacetime, where the distinction between the temporal components of a covariant and contravariant vector becomes important.

The square of the infinitesimal spatial distance between two points in $E^3$ is given by

$$ |d\mathbf{X}|^2 \equiv dX^2 = \delta_{ij}dx^i dx^j = dx^2 + dy^2 + dz^2, \quad (1.9) $$

where $\delta_{ij}$ plays the role of a metric in $E^3$, for an orthonormal basis. The line element $dX^2$ is positive-definite.

### 1.3 Euclidean space isometry group

Requiring coordinate transformations between two inertial frames to leave the spatial ($d\bar{X}^2$) and temporal ($dt^2$) distances unchanged, uniquely determines the form of these transformations. The coordinates in different frames will be distinguished by a bar over the kernel, i.e $\bar{x}^k$. Let us start by showing that the transformation must be linear. Using the chain rule, we have

$$ d\bar{x}^k = \frac{\partial \bar{x}^k}{\partial x^l} dx^l, \quad (1.10) $$

which, imposing the invariance of line element $d\bar{X}^2 = dX^2$, implies

$$ \delta_{ij} = \frac{\partial \bar{x}^k}{\partial x^i} \frac{\partial \bar{x}^l}{\partial x^j} \delta_{kl}. \quad (1.11) $$

Differentiating the previous expression with respect to $x^p$, and taking into account that $\delta_{ij}$ is a constant matrix, we get

$$ \delta_{kl} \left( \frac{\partial^2 \bar{x}^k}{\partial x^p \partial x^j} \frac{\partial \bar{x}^l}{\partial x^i} + \frac{\partial \bar{x}^k}{\partial x^i} \frac{\partial^2 \bar{x}^l}{\partial x^p \partial x^j} \right) = 0. \quad (1.12) $$

Permuting $ipj$ to $pji$ and $jip$ we obtain two equations

$$ \delta_{kl} \left( \frac{\partial^2 \bar{x}^k}{\partial x^p \partial x^j} \frac{\partial \bar{x}^l}{\partial x^i} + \frac{\partial \bar{x}^k}{\partial x^i} \frac{\partial^2 \bar{x}^l}{\partial x^p \partial x^j} \right) = 0, \quad (1.13) $$

$$ \delta_{kl} \left( \frac{\partial^2 \bar{x}^k}{\partial x^p \partial x^i} \frac{\partial \bar{x}^l}{\partial x^j} + \frac{\partial \bar{x}^k}{\partial x^j} \frac{\partial^2 \bar{x}^l}{\partial x^p \partial x^i} \right) = 0. \quad (1.14) $$

Subtracting $(1.13)$ from $(1.12)$, adding $(1.14)$, and taking into account the symmetry of the metric and the fact that the usual derivatives commute, we get

$$ \frac{\partial R^k_i}{\partial x^p} \delta_{kl} R^l_j = 0, \quad (1.15) $$

where we have defined the matrix

$$ R^i_j \equiv \frac{\partial \bar{x}^i}{\partial x^j}. \quad (1.16) $$

Since the transformation $R^i_j$ is required to be have an inverse, we must conclude that

$$ \frac{\partial R^k_i}{\partial x^p} = 0, \quad (1.17) $$

---

3The first index in $R^i_j$ labels rows and the second one labels columns.

4The system $\bar{x}$ is not at all privileged.
which implies that the transformation must be linear

\[
\bar{x}^i = R^i_j x^j + d^i ,
\]

which is nothing else than the indexed version of the orthogonality condition \( R^T R = R^T 1 R = 1 \) for a 3 \( \times \) 3 matrix. \( R^j_i \) is an \( O(3) \) matrix! (as you probably expected). Taking the determinant at both sides of the orthogonality condition, we conclude that the determinant of an orthogonal matrix can take two different values, namely \( \det R = \pm 1 \). Since we will be interested in rotations connected with the identity, we will restrict ourselves to proper rotations with determinant \( \det R = +1 \), i.e. orientation preserving transformations \( SO(3) = \{ R | R^T 1 R = 1, \det R = 1 \} \).

The laws of Newtonian mechanics are required to be covariant, i.e. to have the same form in each inertial frame of reference. In order to achieve so, we will make use of tensors, in this case Cartesian tensors, which have well defined transformation properties from frame to frame. As you will realize soon, these objects are the cornerstone of modern physics theories, such as Special or General Relativity. We will use them repeatedly in this course, so pay attention! We will start our trip using a concrete and familiar context for the introduction of the tensor notions: rotations \( \bar{x}^i = R^i_j x^j \) in Euclidean space.

### 1.4 Tensors in Euclidean space

#### 1.4.1 Scalars

A scalar is single number that does not transform under a coordinate transformations (in this case rotations). Some particular examples of Galilean scalars are the spatial line element (\( dX \)), the temporal line element (\( dt \)), the 3-volume \( d^3 x \equiv |dx dy dz| \), the Lagrangian, the mass of a particle, its charge or any numerical constant.

**Exercise**

Show that the 3-volume is indeed a scalar under rotations.

If we can associate a number to all the points in some spacetime region, as for instance happens with the value of the temperature in the different points of the Earth, we say that we are dealing with a scalar field. Under coordinate transformations, it transforms as

\[
\bar{\phi} (t, \bar{x}) = \phi (t, x) , \quad \text{or} \quad \bar{\phi} (t, \bar{x}) = \phi (t, R^{-1} x) .
\]

#### 1.4.2 Vectors

What is a vector? A vector \( \mathbf{V} \) (in this case Cartesian) is an absolute geometrical object with a particular length and direction which does not depend on the choice of coordinates. The same happens with the rules of vector calculus. Concepts as the angle between two vectors can be defined independently
of the coordinates. Even though there is no need of introducing the concept of components of a vector in a given basis, doing it is sometimes useful. Let us see what happens when we do it. Consider two orthonormal frames related for instance by a rotation of angle \( \theta \) around the \( z \) axis, as illustrated in Fig 1.2. The vector \( \mathbf{V} \) can be expanded in terms of the two set of basis vectors associated to this coordinate systems. In terms of the basis \( \mathbf{e}_i \), the vector \( \mathbf{V} \) has components \( V^i \)

\[
\mathbf{V} = V^i \mathbf{e}_i = V^1 \mathbf{e}_1 + V^2 \mathbf{e}_2 + V^3 \mathbf{e}_3 ,
\]

while, in terms of the rotated basis \( \bar{\mathbf{e}}_i \), it has different components \( \bar{V}^i \)

\[
\mathbf{V} = \bar{V}^i \bar{\mathbf{e}}_i = \bar{V}^1 \bar{\mathbf{e}}_1 + \bar{V}^2 \bar{\mathbf{e}}_2 + \bar{V}^3 \bar{\mathbf{e}}_3 ,
\]

but the vector itself \( \mathbf{V} \) does not change. The relation between the basis vector \( \bar{\mathbf{e}}_i \) and \( \mathbf{e}_i \) can be easily read from the figure to get

\[
\begin{pmatrix}
\bar{e}_1 \\
\bar{e}_2 \\
\bar{e}_3
\end{pmatrix}^T =
\begin{pmatrix}
\mathbf{e}_1 \\
\mathbf{e}_2 \\
\mathbf{e}_3
\end{pmatrix}^T
\begin{pmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix} .
\]

Using this relation, it is easy to write \( \bar{\mathbf{V}} \) in terms of the original basis vectors \( \mathbf{e}_i \) and identify from there the transformation of the components. We obtain

\[
\begin{pmatrix}
\bar{V}_1 \\
\bar{V}_2 \\
\bar{V}_3
\end{pmatrix} =
\begin{pmatrix}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
V_1 \\
V_2 \\
V_3
\end{pmatrix}
\]

The previous exercise can be easily generalized to an arbitrary rotation, giving rise to the following transformation rules

\[
\bar{V}^i = R^i_j V^j ,
\]

\[
\mathbf{e}_i = (R^{-1})^j_i \mathbf{e}_j .
\]

which, in a much powerful notation, can be written as

\[
\bar{V}^i = \frac{\partial \bar{x}^i}{\partial x^j} V^j ,
\]

\[
\mathbf{e}_i = \frac{\partial x^i}{\partial \bar{x}^j} \mathbf{e}_j .
\]

The example has been presented using the \textit{passive viewpoint}, in which the same vector ends up with different components when the reference frame is changed. The expression

\[
\bar{V}^i = R^i_j V^j ,
\]

can also describe the \textit{active viewpoint} in which a given vector is mapped to a different vector under the same basis choice.
In conclusion, a vector $V$ remains unchanged under (in this case) rotations due to the simultaneous and opposite change of its components $V^i$ and the basis $e_i$,

$$V = \bar{V}^i e_i = \left( \frac{\partial \bar{x}^i}{\partial x^j} V^j \right) \left( \frac{\partial e_i}{\partial \bar{x}^k} \right) = V^j \delta^i_j e_k = V^i e_k = V. \quad (1.29)$$

From now on, and in a clear abuse of language, we will frequently employ a standard shorthand and will refer to $V^i$ as a vector instead of saying the components of a vector $V$. A vector is said to be contravariant if it transforms as the displacement vector $dx^i$ (cf. Eq. (1.10))

$$\bar{V}^i = \frac{\partial \bar{x}^i}{\partial x^j} V^j. \quad (1.30)$$

On the other hand, a vector is said to be covariant if it transforms as the basis vectors $e_i$ (cf. Eq. (1.29))

$$\bar{V}_i = \frac{\partial x^i}{\partial \bar{x}^j} V_j. \quad (1.31)$$

A particular example of an object with the previous transformation properties is the gradient of a scalar function

$$\frac{\partial f}{\partial \bar{x}^i} = \frac{\partial x^i}{\partial x^j} \frac{\partial f}{\partial x_j}. \quad (1.32)$$

The gradient is the difference of the function per unit distance in the direction of the basis vector. When the basis vector “shrink” the gradient must “shrink” too.

You maybe think that I am being a bit pedantic here. For you the gradient was, till now, a regular vector, as good as the displacement vector. Now I am giving them two different names and two “different” transformation rules! Indeed...you are right...I am being quite pedantic...but just to prepare the notation for the future. Note the matrix $(R^{-1})^T$, which for the particular case of an orthogonal matrix, is equal to the transformation matrix $R$ itself. As we already said in Section (1.2), there is no clear difference between covariant and contravariant components as long as one transforms between Euclidean orthonormal basis. However, this is not the case in general coordinate systems (such as polar coordinates) or in Special Relativity. Be patient.

Exercises
Show that
- the 3-divergence of a vector field $\partial_i V^i$ transforms as a scalar field.
- the Laplacian operator $\nabla^2 = \partial_i \partial^i$ transforms as a Galilean scalar operator.

1.4.3 Tensors: linear machines

The previous examples are just particular cases of a general class of quantities that transform with a linear and homogeneous transformation law under coordinate transformation: tensors. In order to get some intuition, let us start by considering in detail the transformation laws of rank-2 tensors. In the same way that a vector $V$ can be expanded in terms on the basis $e_i$, a geometric Cartesian tensor $T$ can be expanded as

$$T = T^{ij} e_i \otimes e_j. \quad (1.33)$$
### 1.4 Tensors in Euclidean space

<table>
<thead>
<tr>
<th>Rotations</th>
<th>$\frac{\partial x^i}{\partial x^j} \equiv R^i_j$ are constants!</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scalar</td>
<td>$\bar{\phi} = \phi$</td>
</tr>
<tr>
<td>Contravariant vector</td>
<td>$\bar{V}^i = \frac{\partial x^i}{\partial x^j} V^j$</td>
</tr>
<tr>
<td>Covariant vector</td>
<td>$\bar{V}_i = \frac{\partial x^i}{\partial x^j} V^j$</td>
</tr>
<tr>
<td>Contravariant rank-2 tensor</td>
<td>$\bar{T}^{ij} = \frac{\partial x^i}{\partial x^p} \frac{\partial x^j}{\partial x^q} T^{p,q} $</td>
</tr>
<tr>
<td>Covariant rank-2 tensor</td>
<td>$\bar{T}<em>{ij} = \frac{\partial x^i}{\partial x^p} \frac{\partial x^j}{\partial x^q} T</em>{p,q} $</td>
</tr>
<tr>
<td>Mixed rank-2 tensor</td>
<td>$\bar{T}^i_j = \frac{\partial x^i}{\partial x^p} \frac{\partial x^j}{\partial x^q} T^{p,q} $</td>
</tr>
</tbody>
</table>

Table 1.1

where $\otimes$ denotes the direct product. The transformation property of the different components $T^{ij}$ under a rotation follows immediately from the previous expression: $T^{ij}$ transform as the product of two contravariant vectors $A^i$ and $B^j$

$$\bar{A}^i \bar{B}^j = R^i_k R^j_l A^k B^l \quad \Longrightarrow \quad \bar{T}^{ij} = R^i_k R^j_l T^{kl} .$$

As we did in the previous section, we can define the covariant tensor $T_{ij}$, which transform as the product of two covariant vectors

$$A_i B_j = (R^{-1})^k_i (R^{-1})^l_j A_k B_l \quad \Longrightarrow \quad \bar{T}_{ij} = (R^{-1})^k_i (R^{-1})^l_j T_{kl} .$$

As before, in a clear abuse of language, we will refer to these tensor components as tensors. Particular examples of rank-2 Cartesian tensor are the inertia tensor

$$I^{ij} = \int d^3x \, \rho(x) \left( r^2 \delta^{ij} - x^i x^j \right)$$

or the quadrupole tensor (1.57) (cf. Section 1.5.1).

**Exercise**

Show that the inertia tensor $I^{ij}$ is indeed a rank-(2,0) tensor.

Generalizing the transformation laws (1.34) and (1.35) we can define the transformations properties for arbitrary mixed tensors of contravariant rank $m$ and covariant rank $n$

$$\bar{T}^{i_1 \ldots i_m j_1 \ldots j_n} = \left( \prod_{p=1}^m \frac{\partial x^{i_p}}{\partial x^{k_p}} \prod_{q=1}^n \frac{\partial x^{j_q}}{\partial x^{l_q}} \right) T^{k_1 \ldots k_m l_1 \ldots l_n} \quad \text{(1.37)}$$

$$= \left[ R^{i_1 k_1} \cdots R^{i_m k_m} \right] \left[ (R^{-1})^{j_1 l_1} \cdots (R^{-1})^{j_n l_n} \right] T^{k_1 \ldots k_m l_1 \ldots l_n} \quad \text{(1.38)}$$

Tensors (components) are objects with any number of indices. They share the same transformation properties as vectors and can be classified according to the number of upper or lower indices. For instance, we say that a scalar is a rank-0 tensor and a contravariant (or covariant) vector is a contravariant (or covariant) rank-1 tensor. In general, a tensor with $m$ upper indices and $n$ lower indices is called a rank-$(m,n)$ tensor.
A tensor is not just a quantity carrying indices. It is the transformation law what defines a tensor (see below). Not all quantities with indices are tensors.

1.4.4 Some useful properties

Let me present some useful properties and definitions regarding tensors:

1. The sum (or difference) of two like-tensors is a tensor of the same kind. The proof of this is straightforward. Imagine we take sum or difference of two general tensors $T_{i_1 \ldots i_m j_1 \ldots j_n}$ and $R_{i_1 \ldots i_m j_1 \ldots j_n}$ and apply the transformation rule (1.37), we will get

$$\bar{S}_{i_1 \ldots i_m j_1 \ldots j_n} = \bar{T}_{i_1 \ldots i_m j_1 \ldots j_n} \pm \bar{R}_{i_1 \ldots i_m j_1 \ldots j_n}$$

$$= \left( \prod_{p=1}^{m} \frac{\partial \bar{x}^p}{\partial x^p} \prod_{q=1}^{n} \frac{\partial \bar{x}^q}{\partial x^q} \right) \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial \bar{x}^j}{\partial x^l} \left( \frac{\partial \bar{x}^k}{\partial x^l} \right) = \left( \prod_{p=1}^{m} \frac{\partial \bar{x}^p}{\partial x^p} \prod_{q=1}^{n} \frac{\partial \bar{x}^q}{\partial x^q} \right) \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial \bar{x}^j}{\partial x^l} \left( \frac{\partial \bar{x}^k}{\partial x^l} \right)$$

2. Given two tensors of rank $s$ and $t$, the product transforms as a tensor of rank $(s + t)$.

3. If the expression $T_{i_1 \ldots i_n} = R_{i_1 \ldots i_n} S_{i_1 \ldots i_n}$ is invariant under coordinate transformations and $T_{i_1 \ldots i_n}$ and $R_{i_1 \ldots i_n}$ are tensors, then $S_{i_1 \ldots i_n}$ is a tensor.

Exercise

Prove this for the particular case $T_i = R_j S^j_i$.

4. A tensor contraction occurs when one of a tensor’s free covariant indices is set equal to one of its free contravariant indices. A sum is understood to be performed on the now repeated indices. For instance, $T_{ij} \equiv \sum_{i=1}^{\infty} T_{ij}$ is a contraction on the second and third indices of the tensor $T_{ijk}$.

5. The contraction of a rank-2 tensor is a scalar (its trace) whose value is independent of the coordinate system chosen.

If all the components of a Cartesian tensor $T_{i_1 \ldots i_n}$ in a given inertial reference frame are zero, they will be zero in any other inertial reference frame.

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6Note the words covariant and contravariant. A contraction is never done between two covariant or two contravariant indices.
1.4 Tensors in Euclidean space

1.4.5 Symmetric and antisymmetric tensors

An arbitrary rank-2 tensor can be decomposed into a completely symmetric and a completely antisymmetric part

$$ T_{ij} = T_{(ij)} + T_{[ij]}, $$

(1.40)

where we have used the common notation $(\cdot)$ and $[\cdot]$ to denote respectively symmetrization and antisymmetrization over the indices included inside, i.e.

$$ T_{(ij)} \equiv \frac{1}{2} (T_{ij} + T_{ji}) , \quad T_{[ij]} \equiv \frac{1}{2} (T_{ij} - T_{ji}). $$

(1.41)

Completely symmetric and antisymmetric rank-2 tensors satisfy

$$ T_{ij} = \pm T_{ji}, $$

where the plus sign stands for the symmetric and the minus sign for the antisymmetric one. Particular examples of symmetric tensors are the inertia tensor (1.36) or the quadrupole tensor (1.57) (cf. Section 1.5.1).

Exercise

Prove that the trace of a tensor is invariant under rotations. Show that a tensor $T_{ij}$ in $n$ dimensions has three separately invariant parts

$$ T_{ij} = \frac{1}{n} T^{k} k \delta_{ij} + T_{(ij)} + \left( T_{[ij]} - \frac{1}{n} T^{k} k \delta_{ij} \right). $$

(1.42)

Exercise

Write down the explicit expressions for the completely symmetric and antisymmetric parts of a rank-3 tensor $T_{ijk}$.

1.4.6 Permutation tensor

The Levi-Civita or permutation tensor\(^7\) of rank 3

$$ \epsilon_{ijk} = \epsilon^{ijk} \begin{cases} +1, & \text{if $ijk$ is an even permutation of 123} \\ +1, & \text{if $ijk$ is an odd permutation of 123} \\ 0, & \text{otherwise} \end{cases} $$

(1.43)

flips the sign upon the interchange of any pair of indices and vanishes when two of the indices are equal. Most of the basic identities of vector algebra and vector calculus can be easily proved by using an important relation between the metric tensor $\delta_{ij}$ and $\epsilon_{ijk}$, the *contracted epsilon identity*\(^8\)

$$ \epsilon_{ijk} \epsilon^{lm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}. $$

(1.44)

You will deal with this expression in the exercises.

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\(^7\)Technically, I should say that it is a *pseudotensor*, but we are not interested in introducing this concept here. We will only deal with rotations.

\(^8\)In most of the books you will find this expression with all indices down. Remember that the index convention we chose is just a way of keeping track of the sums that can be easily extended to the Minkowski case. For Cartesian tensors the position of the indices makes no difference.
1.5 Covariance and Classical Mechanics

The main property of tensors is that their transformation law is linear and homogeneous. Each component of a tensor, in this case Cartesian, is a linear combination of the components of the tensor in the original frame, namely

$$T^{i_1...i_m j_1...j_n} = \left( \prod_{p=1}^{m} \frac{\partial \bar{x}^{i_p}}{\partial x^{i_p}} \prod_{q=1}^{n} \frac{\partial x^{l_q}}{\partial \bar{x}^{l_q}} \right) T^{k_1...k_m j_1...j_n}. \quad (1.45)$$

In order to ensure that fundamental equations satisfy the Galilean Principle of Relativity the only thing we have to do is to write tensorial equations. For instance, if two quantities $S^{ij}_k$ and $T^{ij}_k$ transform as rank-(2,1) Cartesian tensors, a fundamental law of the kind

$$S^{ij}_k = T^{ij}_k, \quad (1.46)$$

will retain its form in any inertial reference frame, since both sides of the equation transform in the same way under coordinate transformations (in this case rotations). The fundamental equation (1.46) is then said to be covariant and the transformation is said to be a symmetry of the physical theory.

1.5.1 Newton’s theory of gravity

A physical example of the previous discussion is the Newtonian theory of gravity published by Newton in 1687 within the *Philosophiae Naturalis Principia Mathematica*. In such a theory, the gravitational force $F_i$ exerted on a gravitational test mass $m_G$ is determined by a single function $^9$, the gravitational potential $\Phi$, which depends on the matter distribution through the so-called Poisson equation $^{10}$

$$\nabla^2 \Phi(t,x) = 4\pi G \rho(t,x). \quad (1.49)$$

Eqs. (1.47) and (1.49) are respectively a vector and a scalar covariant equation. If they are valid in a given inertial frame, they will be automatically valid in any inertial frame, since their form will be preserved under rotations and translations.

Exercise: Cosmological constant

Galilean invariance allows for an additional constant $\Lambda$ in the Poisson equation, which becomes

$$\nabla^2 \Phi(t,x) + \Lambda = 4\pi G \rho(t,x). \quad (1.50)$$

Observations of galaxies with typical masses of $10^{30} M_{\odot}$, and intergalactic separations of order 1 Mly do not show any significant deviation from Newton’s inverse square law. Assuming this deviation to be smaller than 1%, determine an upper bound on the magnitude of $\Lambda$.

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$^9$Eqs. (1.47) and (1.49) are left unaltered by the addition to $\Phi$ of an arbitrary function of time $f(t)$, namely

$$\Phi(t,x) \rightarrow \Phi(t,x) + f(t). \quad (1.48)$$

Since the transformation affects only the field $\Phi$ and not the coordinates, the invariance of Eqs. (1.47) and (1.49) under (1.48) is referred as an *internal or gauge symmetry*. The gravitational field $\Phi(t,x)$ has no dynamical degrees of freedom. Eq. (1.49) is not a dynamical equation for the determination of the potential, but rather a constraint on the initial spatial distribution of the potential, which must apply at all times.

$^{10}$No value of the proportionality *Newton’s gravitational constant* $G$ was available to Newton. Its numerical value was firstly determined by Cavendish in 1797 using a torsion balance, being the result reasonably close to present laboratory measurements, $G = 6.674(10) \times 10^{-11} \text{N m}^2/\text{kg}^2$. The gravitational constant remains the most uncertain of all the fundamental constants of physics.
The solution of the Poisson equation can be worked out in the same way that you did for the electromagnetic potential in your Classical Electrodynamics course. The only difference (albeit fundamental) is the sign of the matter distribution. A formal solution of the Poisson’s equation for an arbitrary mass distribution can be obtained by applying the superposition principle or using Green functions to obtain
\[ \Phi(x) = -G \int \frac{\rho(x')}{|x - x'|} \, d^3x', \tag{1.51} \]
where \( x = x'e_i \) is the radius vector of the point at which the gravitational potential is computed, and \( x' = x'^i e_i \) is an arbitrary point in the matter distribution. Note that the Newtonian potential is negative, as expected for an attractive force.

**Exercise: Green’s functions (**)**

Use the Green’s function method to prove Eq.(1.51).

The previous expression becomes the usual \(-GM/r\) only for a spherical mass distribution. The general result for a non-spherical distribution is slightly more complicated. As any distribution function, the essential features of the matter distribution can be be characterized by its moments. For an observer sufficiently far away from the object we can perform a Taylor expansion around \( x' = 0 \) to obtain
\[ \frac{1}{|x - x'|} = e^{-x' \cdot \nabla} \frac{1}{r} = \frac{1}{r} - (x' \cdot \nabla) \frac{1}{r} + \frac{1}{2} (x' \cdot \nabla)^2 \frac{1}{r} + \ldots + \frac{(-1)^n}{n!} (x' \cdot \nabla)^n \frac{1}{r} + \ldots \] (1.52)
\[ = \frac{1}{r} + \frac{x'^k x_k}{r^3} + \frac{(3 x'^k x'^l - r^2 \delta_{kl}) x_k x_l}{2 r^5} + \ldots , \] (1.53)
where we have used the standard expression for the exponential \( e^x = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n \) and defined the distance \( r^2 = x'^k x_k \). Inserted back in Eq.(1.51), we realize that the potential created by the matter distribution
\[ \Phi(x) = -G \left[ \frac{M}{r} + \frac{D^k x_k}{r^3} + \frac{Q^{kl} x_k x_l}{2 r^5} + \ldots \right] , \] (1.54)
can be organized in a series whose individual terms contain information on the spatial structure at an increasing level of detail while decaying the more rapidly in space the higher the information content is. The quantities
\[ M = \int \rho(x') \, d^3x' , \] (1.55)
\[ D^k = \int \rho(x') x'^k \, d^3x' , \] (1.56)
and
\[ Q^{kl} = \int \rho(x') \left( 3 x'^k x'^l - r^2 \delta_{kl} \right) \, d^3x' , \] (1.57)
are respectively the total mass of the system, the mass dipole moment and the mass quadrupole mo-
ment tensor. The dipole moment can be eliminated by simply choosing the origin of coordinates of
the center of mass. The quadrupole moment is the second moment of the mass distribution with its
trace removed. It is proportional to $1/r^3$, which gives rise to a deviation from the inverse square law
of the form $1/r^4$.

**Exercise: Multipole expansion**

- Prove Eq. (1.52).
- Prove that the quadrupole tensor for a spherical distribution vanishes.
- Prove that a change of the origin modifies the quadrupole tensor by only adding a constant.